Spline Modelling of Geostrophic Flow: Theoretical and Algorithmic Aspects

W. Freeden[1], K. Hesse

University of Kaiserslautern
Geomathematics Group
67663 Kaiserslautern
P.O. Box 3049
Germany

phone: ++49 631 205-2852
fax: ++49 631 205-4736
email: freeden@mathematik.uni-kl.de
hesse@mathematik.uni-kl.de
www: http://www.mathematik.uni-kl.de/ wwwgeo

Abstract

Spline functions that approximate (geostrophic) wind field or ocean circulation data are developed in a weighted Sobolev space setting on the (unit) sphere. Two problems are discussed in more detail: the modelling of the (geostrophic) wind field from (i) discrete scalar air pressure data and (ii) discrete vectorial velocity data. Domain decomposition methods based on the Schwarz alternating algorithm for positive definite symmetric matrices are described for solving large linear systems occuring in vectorial spline interpolation or smoothing of geostrophic flow.

Key Words: Geostrophic flow, spherical spline approximation, domain decomposition methods, multiplicative Schwarz alternating algorithm.

[1]Author to whom correspondence should be addressed.

1 Introduction

The atmosphere tends to be close to a state of geostrophic equilibrium at all times. If the velocity v of an element of the atmosphere is specified relative to a reference frame rotating with the (spherical) earth Ω, then the motion is said to be geostrophic when the component of the Coriolis force $-2\,\omega \wedge v$ in the tangential plane of Ω, i.e., $-2\,(\omega \wedge v)_{\mathrm{tan}} = -2(\omega_{\mathrm{tan}} \wedge v_{\mathrm{nor}} + \omega_{\mathrm{nor}} \wedge v_{\mathrm{tan}})$, is balanced by the pressure (surface) gradient across the stream. More explicitly,

$$-2\,\rho(\xi)\,(\omega_{\mathrm{tan}}(\xi) \wedge v_{\mathrm{nor}}(\xi) + \omega_{\mathrm{nor}}(\xi) \wedge v_{\mathrm{tan}}(\xi)) = \nabla^*_\xi P(\xi)\,, \qquad \xi \in \Omega\,,$$

where ρ is the air density and the operator ∇^* represents the tangential (horizontal) surface gradient applied to the scalar pressure field P. If v is assumed to be tangential, i.e., $v_{\mathrm{nor}} = 0$, then the resulting geostrophic balance is given by the well–known equation (see, for example, [22], [23], [24])

$$-2\,\rho(\xi)\,(\omega_{\mathrm{nor}}(\xi) \wedge v_{\mathrm{tan}}(\xi)) = \nabla^*_\xi P(\xi)\,, \quad \xi \in \Omega\,.$$

The normal field ω_{nor} is given by $\omega_{\mathrm{nor}}(\xi) = (\omega \cdot \xi)\,\xi$, $\xi \in \Omega$, with $\omega = |\omega|\,\varepsilon^3$, i.e., the rotation vector ω is supposed to point into the direction of the North Pole ε^3. The geostrophic slope, B, across the stream is given by $B(\xi) := 2\,\omega_{\mathrm{nor}}(\xi) \cdot \xi$, $\xi \in \Omega$, i.e.,

$$B(\xi) = 2\,|\omega|\,(\xi \cdot \varepsilon^3)\,, \quad \xi \in \Omega\,.$$

The resulting equation

$$-B(\xi)\,\rho\,(\xi)\xi \wedge v_{\mathrm{tan}}(\xi) = \nabla^*_\xi P(\xi)\,, \qquad \xi \in \Omega\,,$$

defines the *geostrophic flow*. Using the surface curl gradient $L^*_\xi := \xi \wedge \nabla^*_\xi$ for points $\xi \in \Omega$ we are immediately led to the equation

$$B(\xi)\,\rho(\xi)\,v_{\mathrm{tan}}(\xi) = \xi \wedge \nabla^*_\xi P(\xi) = L^*_\xi P(\xi)\,, \qquad \xi \in \Omega\,, \tag{1}$$

which explicitly gives the description of the *geostrophic velocity* v_{tan}.

Clearly, the geostrophic velocity determined by (1) is perpendicular to the tangential (i.e., horizontal) pressure gradient $\nabla^* P$ on Ω. This is a remarkable feature of the geostrophic flow. The fluid flows along and not across the lines of constant pressure (isobars). It is, indeed, this property that enables the isobars e.g. on a weather map to be representative of the pattern of atmospheric flow.

For geostrophic flow it is worth mentioning that the pressure field can be recovered from the geostrophic velocity v_{tan} by use of the integral formula (i.e., Green's third identity) as developed by [7], [8], [10]

$$P(\xi) = \frac{1}{4\pi}\int_\Omega P(\eta)\,d\omega(\eta) - \int_\Omega L^*_\eta G(\Delta^*;\xi,\eta)\,L^*_\eta P(\eta)\,d\omega(\eta)\,, \tag{2}$$

where $G(\Delta^*;\cdot,\cdot)$ denotes the Green function with respect to the Beltrami operator Δ^* given by

$$G(\Delta^*;\xi,\eta) = \frac{1}{4\pi}\,\ln\,(1-\xi\cdot\eta) + \frac{1}{4\pi} - \frac{1}{4\pi}\,\ln(2)\,,$$

$(\xi, \eta) \in \Omega \times \Omega$, $-1 \leq \xi \cdot \eta < 1$. In other words, assuming that the mean value P_{mean}, which is given by $P_{\mathrm{mean}} = \frac{1}{4\pi} \int_\Omega P(\eta)\, d\omega(\eta)$, is known we obtain

$$P(\xi) - P_{\mathrm{mean}} = -\int_\Omega L_\eta^* G(\Delta^*; \xi, \eta)\, B(\eta)\, \rho(\eta)\, v_{\mathrm{tan}}(\eta)\, d\omega(\eta)\,. \tag{3}$$

If, in addition, the air density is replaced by the mean value ρ_{mean} we find

$$P(\xi) - P_{\mathrm{mean}} = \frac{\rho_{\mathrm{mean}}|\omega|}{2\pi} \int_\Omega \frac{\xi \wedge \eta}{1 - \xi \cdot \eta}\, (\eta \cdot \varepsilon^3)\, v_{\mathrm{tan}}(\eta)\, d\omega(\eta)\,,$$

which provides an integral relation for deriving pressure information from the geostrophic velocity v_{tan}.

In this paper we are concerned with the following (discrete) problems of geostrophic flow in the atmosphere:

- **Pressure Data Problem (PDP):** Let there be known the scalar pressure field P for a finite subset $\{\xi_1^N, \ldots, \xi_N^N\}$ of points on the (unit) sphere Ω. Find a smooth approximation of P (and via (1) of v_{tan}) from the discrete data

  $$\left\{ \left(\xi_i^N, P(\xi_i^N) + \epsilon_i\right) \middle|\ i = 1, \ldots, N \right\}\,,$$

  where $\epsilon_1, \ldots, \epsilon_N$ are (scalar) measurement errors.

- **Wind Data Problem (WDP):** Let there be known the vectorial wind field v_{tan} for a finite subset $\{\xi_1^N, \ldots, \xi_N^N\}$ of points on the unit sphere. Find a smooth approximation of v_{tan} (and via (3) of $P - P_{\mathrm{mean}}$) from the discrete data

  $$\left\{ \left(\xi_i^N, B(\xi_i^N)\, \rho(\xi_i^N)\, v_{\mathrm{tan}}(\xi_i^N) + \epsilon_i\right) \middle|\ i = 1, \ldots, N \right\},$$

  where $\epsilon_1, \ldots, \epsilon_N$ are (vectorial) measurement errors.

It should be remarked that in the same way the atmospheric flows are derived from the surface pressure field, P, upper ocean flow can be determined from the knowledge of the sea surface dynamic topography Ξ, i.e., the difference between the sea surface height and the geoid (cf. [23], [24]). In other words, the pressure gradients are revealed as the deviations of the sea surface from an equipotential surface like the geoid such that

$$B(\xi)\, \rho(\xi)\, v_{\mathrm{tan}}(\xi) = L_\xi^* \Xi(\xi)\,, \qquad \xi \in \Omega\,.$$

Obviously, the observed ocean currents follow the contours of the dynamic topography, and this fact provides a check on the validity of both the mean surface maps and the geostrophic assumption (see [18], [23], [26] and many others). Correspondingly, PDP can also be understood as sea surface dynamic topography data problem, while WDP may be interpreted as ocean circulation data problem.

The significance of our considerations lies in the development of a general framework in which it is possible that the approximation may be chosen to embody desirable characteristics in accordance with the source of the data. This feature is achieved through the idea of approximation by splines that minimize a weighted Sobolev norm (energy norm), with a large class of weights being at the disposal of the user. The Sobolev space, along the variational properties of spline interpolation or spline smoothing yield some important benefits: The spherical splines we need here are constituted as linear combinations of (scalar or vectorial) radial basis functions (see [10], [4]) which can be recognized as reproducing kernels functions associated to the Sobolev spaces. According to this construction, huge amounts of data can be handled by a domain decomposition method within the solution process of the resulting linear equations, the coefficient matrix of them being of Gram type. Because of their space localizing properties, the constituting kernel functions are of great importance, in particular, when local approximants are required (such as spline approximants of wind fields on local areas or ocean currents on parts of the see surface).

In oceanography the pressure field P on Ω is conventionally represented as a Fourier expansion in terms of spherical harmonics. This approach consequently leads to a series expansion of the velocity field in terms of (divergence-free) vector spherical harmonics (see, for example, [18], [22], [23]). However, these vector types of functions are far from being suitable for purposes of local approximation. In the case of ocean circulation, boundary effects like the Gibbs phenomenon cannot be avoided by non-space-localizing spherical harmonics when the dynamic topography is set to zero over the continents. By use of space-localizing spline functions as proposed here no assumption on the continents must be made, and the approximation along the coast lines can be handled in a much better way.

The layout of the paper is as follows: First background material is given. Vector spherical harmonics are introduced in a standard way. Then the characterization of spline approximation in a weighted Sobolev space setting is described for the problems PDP and WDP, respectively. It is shown how certain choices of weight sequences yield well-suited approximants in terms of elementary available and easily implementable kernel expressions. In order to solve linear systems involved in (spherical) spline approximation problems a domain decomposition method based on the Schwarz alternating algorithm is explained in detail. Some remarks on the numerical implementation for the spline modelling of geostrophic wind fields conclude the paper.

2 Preliminaries

We begin our consideration by introducing some basic notation that will be used throughout the paper.

2.1 Notation

Let $\mathbb{N}$, $\mathbb{R}$ denote the positive integers and the real numbers, respectively. Furthermore, we set $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$, and $\mathbb{R}^+ := \{t \in \mathbb{R} \,|\, t > 0\}$. We denote the Euclidean inner product on $\mathbb{R}^n$ by $(x, y) = x \cdot y := \sum_{j=1}^{n} x_j\, y_j$, and the Euclidean norm is designated by $|x| := \sqrt{(x, x)}$, where $x = (x_1, \ldots, x_n)^T$, $y = (y_1, \ldots, y_n)^T$. For all $x \in \mathbb{R}^3$, $x = (x_1, x_2, x_3)^T$, different from the origin, we have

$$x = r\,\xi\,, \quad r = |x| = \sqrt{x_1^2 + x_2^2 + x_3^2}\,, \tag{4}$$

where $\xi = (\xi_1, \xi_2, \xi_3)^T$ is the uniquely determined directional unit vector of $x \in \mathbb{R}^3$. The unit sphere in $\mathbb{R}^3$ will be denoted by Ω. If the vectors $\varepsilon^1, \varepsilon^2, \varepsilon^3$ form the canonical orthonormal basis in $\mathbb{R}^3$, we may represent the points $\xi \in \Omega$ in spherical coordinates by

$$\xi = t\,\varepsilon^3 + \sqrt{1-t^2}\left(\cos(\varphi)\,\varepsilon^1 + \sin(\varphi)\,\varepsilon^2\right) , \tag{5}$$

where $-1 \le t \le 1$, $0 \le \varphi < 2\pi$, $t = \cos(\vartheta)$. Apart from the poles (i.e., $t \in \{-1, 1\}$), the vectors

$$\begin{aligned}
\varepsilon^\varphi = \varepsilon^\varphi(\varphi, t) &:= \frac{\partial \xi(\varphi,t)}{\partial \varphi} \left| \frac{\partial \xi(\varphi,t)}{\partial \varphi} \right|^{-1} = \begin{pmatrix} -\sin(\varphi) \\ \cos(\varphi) \\ 0 \end{pmatrix} , \\
\varepsilon^t = \varepsilon^t(\varphi, t) &:= \frac{\partial \xi(\varphi,t)}{\partial t} \left| \frac{\partial \xi(\varphi,t)}{\partial t} \right|^{-1} = \begin{pmatrix} -t\,\cos(\varphi) \\ -t\,\sin(\varphi) \\ \sqrt{1-t^2} \end{pmatrix}
\end{aligned}$$

form a basis of the tangential space to Ω in the point $\xi = \xi(\varphi, t)$.

As usual, the vector product of two vectors $x, y \in \mathbb{R}^3$ is defined by

$$x \wedge y := (x_2\,y_3 - x_3\,y_2, x_3\,y_1 - x_1\,y_3, x_1\,y_2 - x_2\,y_1)^T .$$

In terms of the coordinates (4) the gradient ∇ in $\mathbb{R}^3$ reads as follows

$$\nabla_x = \xi\,\frac{\partial}{\partial r} + \frac{1}{r}\,\nabla^*_\xi ,$$

where ∇^* is the (aforementioned) *surface gradient* of the unit sphere $\Omega \subset \mathbb{R}^3$. Moreover, the *Laplace operator* $\Delta := \nabla \cdot \nabla$ in $\mathbb{R}^3$ allows the representation

$$\Delta_x = \left(\frac{\partial}{\partial r}\right)^2 + \frac{2}{r}\,\frac{\partial}{\partial r} + \frac{1}{r^2}\,\Delta^*_\xi ,$$

where $\Delta^* = \nabla^* \cdot \nabla^*$ is the *Beltrami operator* of the unit sphere Ω (for explicit representations in terms of the coordinates (5) see, e.g., [10]).

Throughout this paper scalar-valued (vector-valued) functions are denoted by capital (small) letters. A function $F : \Omega \to \mathbb{R}$ ($f : \Omega \to \mathbb{R}^3$) possessing k continuous derivatives on the unit sphere Ω is said to be of class $\mathcal{C}^{(k)}(\Omega)$ ($c^{(k)}(\Omega)$). $\mathcal{C}^{(0)}(\Omega) = \mathcal{C}(\Omega)$ ($c^{(0)}(\Omega) = c(\Omega)$) is the class of real continuous scalar-valued (vector-valued) functions on Ω.

For $F \in C^{(1)}(\Omega)$ we introduce the *surface curl gradient* L^* via

$$L^*_\xi F(\xi) := \xi \wedge \nabla^*_\xi F(\xi) , \qquad \xi \in \Omega .$$

Furthermore, $\nabla^*_\xi \cdot f(\xi)$, $\xi \in \Omega$, and $L^*_\xi \cdot f(\xi)$, $\xi \in \Omega$, respectively, denote the *surface divergence* and the *surface curl* of the vector field f at $\xi \in \Omega$. (It should be noted that the operators ∇^*, L^*, Δ^* will be always used in coordinate-free representation throughout this work, thereby avoiding any singularities at the poles).

The operators $o^{(i)} : C^{(1)}(\Omega) \to c(\Omega)$, $i = 1, 2, 3$, defined by

$$\begin{aligned}
o^{(1)}_\xi F(\xi) &:= \xi F(\xi) , \qquad \xi \in \Omega , \\
o^{(2)}_\xi F(\xi) &:= \nabla^*_\xi F(\xi) , \qquad \xi \in \Omega , \\
o^{(3)}_\xi F(\xi) &:= L^*_\xi F(\xi) , \qquad \xi \in \Omega ,
\end{aligned}$$

are of particular importance for our considerations. Therefore, we list some of their properties in more detail. For all $\xi \in \Omega$ we have

$$o_\xi^{(i)} F(\xi) \cdot o_\xi^{(j)} F(\xi) = 0$$

whenever $j \neq i$, $i, j \in \{1, 2, 3\}$. Moreover, if $G \in C^{(1)}[-1, +1]$ and $(\xi, \eta) \in \Omega \times \Omega$, it is not hard to see that

$$\begin{aligned} o_\xi^{(2)} G(\xi \cdot \eta) &= G'(\xi \cdot \eta)\,(\eta - (\xi \cdot \eta)\xi)\,, \\ o_\xi^{(3)} G(\xi \cdot \eta) &= G'(\xi \cdot \eta)\,(\xi \wedge \eta)\,. \end{aligned} \tag{6}$$

Furthermore, Green's identities show us that (cf. (2))

$$F(\xi) = \frac{1}{4\pi} \int_\Omega F(\eta)\, d\omega(\eta) - \int_\Omega \left(o_\eta^{(i)} G(\Delta^*; \xi, \eta) \right) \cdot \left(o_\eta^{(i)} F(\eta) \right) d\omega(\eta)$$

holds for all $\xi \in \Omega$, $i \in \{2, 3\}$, and $F \in C^{(1)}(\Omega)$, where $G(\Delta^*; \cdot, \cdot)$ is the Green function with respect to the Beltrami operator Δ^* (cf. [7], [8]). The integral formulas (cf. [10]) for $F \in \mathcal{C}^{(1)}(\Omega)$, $f \in c^{(1)}(\Omega)$

$$\begin{aligned} \int_\Omega f(\xi) \cdot \nabla_\xi^* F(\xi)\, d\omega(\eta) &= -\int_\Omega F(\xi)\, \nabla_\xi^* \cdot f(\xi)\, d\omega(\eta)\,, \\ \int_\Omega f(\xi) \cdot L_\xi^* F(\xi)\, d\omega(\eta) &= -\int_\Omega F(\xi)\, L_\xi^* \cdot f(\xi)\, d\omega(\eta) \end{aligned}$$

lead us to operators $O^{(i)} : c^{(1)}(\Omega) \to C^{(0)}(\Omega)$, $i = 1, 2, 3$, which are adjoint to $o^{(i)}$. To be more concrete, for $f \in c^{(1)}(\Omega)$ and $F \in C^{(1)}(\Omega)$, we have

$$\int_\Omega o_\xi^{(i)} F(\xi) \cdot f(\xi)\, d\omega(\eta) = \int_\Omega F(\xi)\, O_\xi^{(i)} f(\xi)\, d\omega(\eta)\,, \qquad i = 1, 2, 3\,,$$

where

$$\begin{aligned} O_\xi^{(1)} f(\xi) &:= \xi \cdot p_{\mathrm{nor}} f(\xi)\,, & \xi \in \Omega\,, \\ O_\xi^{(2)} f(\xi) &:= -\nabla_\xi^* \cdot p_{\tan} f(\xi)\,, & \xi \in \Omega\,, \\ O_\xi^{(3)} f(\xi) &:= -L_\xi^* \cdot p_{\tan} f(\xi)\,, & \xi \in \Omega\,, \end{aligned}$$

and $p_{\mathrm{nor}} f(\xi) := (f(\xi) \cdot \xi)\, \xi$, $p_{\tan} f(\xi) := f(\xi) - p_{\mathrm{nor}} f(\xi)$, $\xi \in \Omega$. It can be easily seen that

$$O_\xi^{(i)} o_\xi^{(j)} F(\xi) = 0\,, \qquad i \neq j, \quad i, j \in \{1, 2, 3\}\,,$$

and

$$O^{(i)} o^{(i)} F(\xi) = \begin{cases} F(\xi) & \text{if} \quad i = 1 \\ -\Delta_\xi^* F(\xi) & \text{if} \quad i = 2, 3\,, \end{cases}$$

provided that F is of class $C^{(2)}(\Omega)$. For more details the reader is referred to [2], [10].

By $l^2(\Omega)$ we denote the space of (Lebesgue) square-integrable vector fields on Ω, i.e.,

$$l^2(\Omega) := \left\{ f : \Omega \to \mathbb{R}^3 \,\middle|\, \int_\Omega f(\xi) \cdot f(\xi)\, d\omega(\xi) < \infty \right\}.$$

$l^2(\Omega)$ is a Hilbert space equipped with the inner product

$$(f,g)_{l^2(\Omega)} = \left(\int_\Omega f(\xi) \cdot g(\xi)\, d\omega(\xi) \right)^{1/2} .$$

For a given vector field $f : \Omega \to \mathbb{R}^3$

$$p_{\mathrm{nor}} f : \xi \mapsto p_{\mathrm{nor}} f(\xi) := (\xi \cdot f(\xi))\, \xi\,, \qquad \xi \in \Omega\,,$$

is called the *normal part* of f, while

$$p_{\tan} f : \xi \mapsto p_{\tan} f(\xi) := f(\xi) - p_{\mathrm{nor}} f(\xi)\,, \qquad \xi \in \Omega\,,$$

is called the *tangential part* of f. A vector field $f : \Omega \to \mathbb{R}^3$ is called *tangential* (*normal*), if $f(\xi) = p_{\tan} f(\xi)$ $(f(\xi) = p_{\mathrm{nor}} f(\xi))$ for all $\xi \in \Omega$.

Analogously, we denote by $\mathcal{L}^2(\Omega)$ the Hilbert space of (Lebesgue) square-integrable scalar functions with the inner product

$$(F,G)_{\mathcal{L}^2(\Omega)} := \int_\Omega F(\xi)\, G(\xi)\, d\omega(\xi)\,.$$

The study of vector fields on the sphere can be greatly simplified by the so-called *Helmholtz decomposition theorem* for continuously differentiable vector fields $f : \Omega \to \mathbb{R}^3$ (see [10])

$$f(\xi) = p_{\mathrm{nor}} f(\xi) + p_{\tan} f(\xi)\,, \qquad \xi \in \Omega\,.$$

To be more precise, any continuously differentiable vector field on the unit sphere $\Omega \subset \mathbb{R}^3$ (i.e., $f \in c^{(1)}(\Omega)$) may be represented by a decomposition in terms of scalar functions $F^{(i)} \in C^{1+0_i}(\Omega)$, $i = 1,2,3$, where $0_1 := 0$, $0_2 := 1$, $0_3 := 1$, such that

$$\begin{aligned} p_{\mathrm{nor}} f(\xi) &= o_\xi^{(1)} F^{(1)}(\xi)\,, \qquad \xi \in \Omega\,, \\ p_{\tan} f(\xi) &= o_\xi^{(2)} F^{(2)}(\xi) + o_\xi^{(3)} F^{(3)}(\xi)\,, \qquad \xi \in \Omega\,, \end{aligned}$$

where

$$\begin{aligned} F^{(1)}(\xi) &= \xi \cdot f(\xi)\,, \qquad \xi \in \Omega\,, \\ F^{(2)}(\xi) &= -\int_\Omega G(\Delta^*;\xi,\eta)\, O_\eta^{(2)} p_{\tan} f(\xi)\, d\omega(\eta)\,, \qquad \xi \in \Omega\,, \\ F^{(3)}(\xi) &= -\int_\Omega G(\Delta^*;\xi,\eta)\, O_\eta^{(3)} p_{\tan} f(\xi)\, d\omega(\eta)\,, \qquad \xi \in \Omega\,. \end{aligned}$$

These representations of spherical vector fields lay the groundwork for the main subject of this work. The explicit representations of the tangential as well as the normal field, in fact, are essential for the constructive approximation of spherical vector fields.

2.2 Vector Spherical Harmonics

After these preparations we are able to introduce vector spherical harmonics.

Definition 2.1 *Any vector field $o^{(i)}Y_n$, $i \in \{1,2,3\}$, where Y_n is a spherical harmonic of order $n \geq 0_i$, is called a vector spherical harmonic of order n and type i.*

Assume that $\{Y_{n,j}\}_{n\in\mathbb{N}_0;\, j=1,\ldots,2n+1}$ is a complete orthonormal system of spherical harmonics in $(\mathcal{L}^2(\Omega), (\cdot,\cdot)_{\mathcal{L}^2(\Omega)})$. Then, for $i \in \{1,2,3\}$, we define the system $\{y^{(i)}_{n,j}\}_{n\in\mathbb{N}_{0_i};\, j=1,\ldots,2n+1}$, by

$$y^{(i)}_{n,j} := \left(\mu^{(i)}_n\right)^{-1/2} o^{(i)}Y_{n,j}\,, \tag{7}$$

where $\mathbb{N}_{0_i} = \mathbb{N}_0$ for $i=1$ and $\mathbb{N}_{0_i} = \mathbb{N}$ for $i = 2,3$ and

$$\mu^{(i)}_n := \begin{cases} 1 & \text{if} \quad i=1 \\ n(n+1) & \text{if} \quad i = 2,3\,. \end{cases}$$

It is not difficult to show that

$$\begin{aligned}
\int_\Omega y^{(i)}_{n,j}(\xi)\cdot y^{(k)}_{m,l}(\xi)\,d\omega(\xi) &= (\mu^{(i)}_n)^{-1/2}(\mu^{(k)}_m)^{-1/2}\int_\Omega Y_{n,j}(\xi)\,O^{(i)}_\xi o^{(k)}_\xi Y_{m,l}(\xi)d\omega(\xi) \\
&= \delta_{i,k}\,(\mu^{(i)}_n)^{-1/2}(\mu^{(k)}_m)^{-1/2}\int_\Omega Y_{n,j}(\xi)\left(\mu^{(i)}_m\,Y_{m,l}(\xi)\right)d\omega(\xi) \\
&= \delta_{ik}\,\delta_{nm}\,\delta_{jl}\,.
\end{aligned}$$

Hence, the set

$$\left\{ y^{(i)}_{n,j} \;\middle|\; n\in\mathbb{N}_{0_i};\, j=1,\ldots,2n+1;\, i\in\{1,2,3\}\right\}$$

is an orthonormal system in the space $(l^2(\Omega), (\cdot,\cdot)_{l^2(\Omega)})$.

The next result states that this system is closed and complete in $l^2(\Omega)$.

Theorem 2.2 *Let $\{Y_{n,j}\}_{n\in\mathbb{N}_0;\, j=1,\ldots,2n+1}$ be a complete orthonormal system of spherical harmonics in $\mathcal{L}^2(\Omega)$. Then the set $\{y^{(i)}_{n,j}\,|\, n\in\mathbb{N}_{0_i}; j=1,\ldots,2n+1; i\in\{1,2,3\}\}$, defined by (7), shows the following properties:*

(i) $\overline{\operatorname{span}\left\{ y^{(i)}_{n,j} \;\middle|\; n\in\mathbb{N}_{0_i}; j=1,\ldots,2n+1; i\in\{1,2,3\}\right\}}^{\|\cdot\|_{c(\Omega)}} = c(\Omega)\,.$

(ii) $\left\{ y^{(i)}_{n,j} \;\middle|\; n\in\mathbb{N}_{0_i}; j=1,\ldots,2n+1; i\in\{1,2,3\}\right\}$ *is a complete orthonormal system in the Hilbert space $(l^2(\Omega); (\cdot,\cdot)_{l^2(\Omega)})$.*

Proof. The proof of this theorem can be found in [10]. ■

In particular, Theorem 2.2 implies that any $f \in l^2(\Omega)$ has a Fourier series expansion of the form

$$f = \sum_{i=1}^{3}\sum_{n=0_i}^{\infty}\sum_{j=1}^{2n+1}(f^{(i)})^\wedge(n,j)\,y^{(i)}_{n,j}\,, \tag{8}$$

where the Fourier coefficients $(f^{(i)})^\wedge(n,j)$, $n\in\mathbb{N}_{0_i}$; $j=1,\ldots,2n+1$; $i\in\{1,2,3\}$, are defined by

$$(f^{(i)})^\wedge(n,j) := \int_\Omega f(\xi)\cdot y^{(i)}_{n,j}(\xi)\,d\omega(\xi).$$

Of course, the convergence of the series expansion (8) of the vector field $f \in l^2(\Omega)$ is understood in $l^2(\Omega)$-sense.

The space $l^2(\Omega)$ can be decomposed as follows:

$$l^2(\Omega) = l^2_{(1)}(\Omega) \oplus l^2_{(2)}(\Omega) \oplus l^2_{(3)}(\Omega)\,,$$

where for $i = 1, 2, 3$

$$l^2_{(i)}(\Omega) := \overline{\operatorname{span}\left\{ y^{(i)}_{n,j} \,\middle|\, n \in \mathbb{N}_{0_i};\ j = 1, \ldots, 2n+1 \right\}}^{\|\cdot\|_{l^2(\Omega)}}.$$

Moreover, we have

$$c(\Omega) = c_{(1)}(\Omega) \oplus c_{(2)}(\Omega) \oplus c_{(3)}(\Omega)\,,$$

where for $i \in \{1, 2, 3\}$

$$c_{(i)}(\Omega) := \overline{\operatorname{span}\left\{ y^{(i)}_{n,j} \,\middle|\, n \in \mathbb{N}_{0_i};\ j = 1, \ldots, 2n+1 \right\}}^{\|\cdot\|_{c(\Omega)}}.$$

Furthermore, we define for $k \in \mathbb{N} \cup \{\infty\}$

$$c^{(k)}_{(i)}(\Omega) := c_{(i)}(\Omega) \cap c^{(k)}(\Omega), \qquad i \in \{1, 2, 3\}\,.$$

Finally, we mention the following result for vector spherical harmonics:

Lemma 2.3 *Let $\{Y_{n,j}\}_{n \in \mathbb{N}_0;\, j=1,\ldots,2n+1}$ be a complete orthonormal system of spherical harmonics in $\mathcal{L}^2(\Omega)$. Set $y^{(i)}_{n,j} := (\mu^{(i)}_n)^{-1/2} o^{(i)} Y_{n,j}, n \in \mathbb{N}_{0_i};\ j = 1, \ldots, 2n+1;\ i \in \{1, 2, 3\}$. Then, for $i \in \{1, 2, 3\}$,*

$$\sum_{j=1}^{2n+1} \left| y^{(i)}_{n,j}(\xi) \right|^2 = \frac{(2n+1)}{4\pi}\,, \qquad \xi \in \Omega,\ n \in \mathbb{N}_{0_i}\,.$$

Proof. The proof follows from the addition theorem of vector spherical harmonics that can be found in [2], [10]. ■

From now on, we make the convention (without further mentioning) that $y^{(i)}_{n,j}$ is always defined by (7) by use of a complete orthonormal system $\{Y_{n,j}\}_{n \in \mathbb{N}_0;\, j=1,\ldots,2n+1}$ of spherical harmonics in $\mathcal{L}^2(\Omega)$.

3 Sobolev-Like Hilbert Spaces or Scalar- or Vector-Valued Functions on the Sphere

Next Sobolev-like Hilbert spaces, which are subspaces of $\mathcal{L}^2(\Omega)$ or $l^2_{(i)}(\Omega)$, $i \in \{1, 2, 3\}$, respectively, are introduced. Their properties are summed up. After that, it is explained under which conditions certain linear operators on these spaces, which are important in geostrophic wind field determination, are bounded. Finally, the representation of bounded linear functionals on these spaces is discussed. Some types of bounded linear functionals (which are relevant for this publication) are investigated in more detail.

3.1 Sobolev-like Subspaces of $\mathcal{L}^2(\Omega)$

The definition of the (scalar) Sobolev-like subspaces of $\mathcal{L}^2(\Omega)$ is in analogy to the approach presented in [8], where more details can be found. The Sobolev-like subspaces of $\mathcal{L}^2(\Omega)$ are useful for the understanding of the respective subspaces of $l^2_{(i)}(\Omega)$.

Definition 3.1 *Let $\{A_n\}_{n\in\mathbb{N}_0} \subset \mathbb{R}^+$ satisfy $A_n \geq C$ for all $n \in \mathbb{N}_0$ with some constant $C > 0$. We introduce the set*

$$\mathcal{E}(\{A_n\};\Omega) := \left\{ F \in C^{(\infty)}(\Omega) \,\middle|\, \sum_{n=0}^{\infty} \sum_{j=1}^{2n+1} A_n^2 \left(F^\wedge(n,j)\right)^2 < \infty \right\},$$

where $\{F^\wedge(n,j)\}_{n\in\mathbb{N}_0;\, j=1,\ldots,2n+1}$ is the set of Fourier coefficients of F with respect to the complete orthonormal system $\{Y_{n,j}\}_{n\in\mathbb{N}_0;\, j=1,\ldots,2n+1}$ of spherical harmonics in $\mathcal{L}^2(\Omega)$, i.e.,

$$F^\wedge(n,j) = \int_\Omega F(\xi)\, Y_{n,j}(\xi)\, d\omega(\xi)\,.$$

On $\mathcal{E}(\{A_n\};\Omega)$ we impose the inner product

$$(F,G)_{\mathcal{H}(\{A_n\};\Omega)} := \sum_{n=0}^{\infty} \sum_{j=1}^{2n+1} A_n^2\, F^\wedge(n,j)\, G^\wedge(n,j)\,,$$

which induces the norm

$$\|F\|_{\mathcal{H}(\{A_n\};\Omega)} := \left(\sum_{n=0}^{\infty} \sum_{j=1}^{2n+1} A_n^2 \left(F^\wedge(n,j)\right)^2 \right)^{1/2}.$$

The Sobolev-like Hilbert space (Sobolev space) $\mathcal{H}(\{A_n\};\Omega)$ is defined to be the completion

$$\mathcal{H}(\{A_n\};\Omega) := \overline{\mathcal{E}(\{A_n\};\Omega)}^{\|\cdot\|_{\mathcal{H}(\{A_n\};\Omega)}}\,.$$

A complete orthonormal system in $\mathcal{H}(\{A_n\};\Omega)$ is given by $\left\{A_n^{-1}Y_{n,j}\right\}_{n\in\mathbb{N}_0;\, j=1,\ldots,2n+1}$. Due to the estimate

$$\begin{aligned}
\|F\|^2_{\mathcal{L}^2(\Omega)} &= \frac{1}{C^2}\left(\sum_{n=0}^{\infty} \sum_{j=1}^{2n+1} C^2 \left(F^\wedge(n,j)\right)^2 \right) \\
&\leq \frac{1}{C^2}\left(\sum_{n=0}^{\infty} \sum_{j=1}^{2n+1} A_n^2 \left(F^\wedge(n,j)\right)^2 \right) \\
&= \frac{1}{C^2}\, \|F\|^2_{\mathcal{H}(\{A_n\};\Omega)}
\end{aligned}$$

for $F \in \mathcal{E}(\{A_n\};\Omega)$, we are able to deduce that every Cauchy sequence in $\mathcal{E}(\{A_n\};\Omega)$ with respect to $\|\cdot\|_{\mathcal{H}(\{A_n\};\Omega)}$ is also a Cauchy sequence with respect to $\|\cdot\|_{\mathcal{L}^2(\Omega)}$. Thus, $\mathcal{H}(\{A_n\};\Omega)$ is a subspace of $\mathcal{L}^2(\Omega)$. Every $F \in \mathcal{H}(\{A_n\};\Omega)$ can be represented by the series expansion

$$F = \sum_{n=0}^{\infty} \sum_{j=1}^{2n+1} F^\wedge(n,j)\, Y_{n,j}\,, \tag{9}$$

where $F^\wedge(n,j)$ is, as before, the Fourier coefficient $F^\wedge(n,j) = \int_\Omega F(\xi)\, Y_{n,j}(\xi)\, d\omega(\xi)$. The series expansion (9) of $F \in \mathcal{H}(\{A_n\};\Omega)$ converges both in the $\mathcal{L}^2(\Omega)$-sense and with respect to $\|\cdot\|_{\mathcal{H}(\{A_n\};\Omega)}$. It is clear that $\mathcal{H}(\{A_n\};\Omega)$ is a dense subset of $\mathcal{L}^2(\Omega)$, because $\mathcal{H}(\{A_n\};\Omega)$ obviously contains all spherical harmonics $Y_{n,j}$, $n \in \mathbb{N}_0; j = 1,\ldots,2n+1$.

Theorem 3.2 *Let $\{A_n\}_{n\in\mathbb{N}_0} \subset \mathbb{R}^+$ satisfy $A_n \geq C$ for all $n \in \mathbb{N}_0$ with a constant $C > 0$, and let $\mathcal{H}(\{A_n\};\Omega)$ be defined as indicated by Definition 3.1. Then the following statements are valid:*

(i) If $\sum_{n=0}^\infty \frac{(n+\frac{1}{2})}{A_n^2} < \infty$, then $\mathcal{H}(\{A_n\};\Omega) \subset \mathcal{C}(\Omega)$.

(ii) If $\sum_{n=0}^\infty \frac{(n+\frac{1}{2})^3}{A_n^2} < \infty$, then $\mathcal{H}(\{A_n\};\Omega) \subset \mathcal{C}^{(1)}(\Omega)$. The series expansion (9) of a function $F \in \mathcal{H}(\{A_n\};\Omega)$ may be differentiated term by term.

Proof. This theorem is also known as (the spherical counterpart of the) Sobolev Lemma. A proof of (i) can be found in, for example, [9]. Statement (ii) follows from the well-known theorem of analysis about the differentiation of sequences of functions, where the main trick is to parameterize the truncated Fourier series expansion locally in spherical coordinates and to express the directional derivatives with respect to the polar coordinates with the help of L^* and ∇^*. Lemma 2.3 then yields all estimates that are needed to ensure the term by term differentiation of (9). ∎

Finally, we note that $\mathcal{H}(\{A_n\};\Omega)$ is a reproducing kernel Hilbert space if and only if

$$\sum_{n=0}^\infty \frac{(n+\frac{1}{2})}{A_n^2} < \infty. \tag{10}$$

For a proof see, for example, [8] or [10].

3.2 Sobolev-like Subspaces of $l^2_{(i)}(\Omega)$

Next we define subspaces of $l^2_{(i)}(\Omega)$ in analogy to Definition 3.1.

Definition 3.3 *Let $\{A_n\}_{n\in\mathbb{N}_0} \subset \mathbb{R}^+$ be a sequence of positive numbers satisfying $A_n \geq C$ for all $n \in \mathbb{N}_0$ with some constant $C > 0$. For $i \in \{1,2,3\}$ we let*

$$\varepsilon^{(i)}(\{A_n\};\Omega) := \left\{ f \in c^{(\infty)}_{(i)}(\Omega) \;\middle|\; \sum_{n=0_i}^\infty \sum_{j=1}^{2n+1} A_n^2 \left((f^{(i)})^\wedge(n,j) \right)^2 < \infty \right\}.$$

On $\varepsilon^{(i)}(\{A_n\};\Omega)$ we impose the inner product

$$(f,g)_{h^{(i)}(\{A_n\};\Omega)} = \sum_{n=0_i}^\infty \sum_{j=1}^{2n+1} A_n^2\, (f^{(i)})^\wedge(n,j)\, (g^{(i)})^\wedge(n,j)\,,$$

which induces the norm

$$\|f\|_{h^{(i)}(\{A_n\};\Omega)} := \left(\sum_{n=0_i}^\infty \sum_{j=1}^{2n+1} A_n^2 \left((f^{(i)})^\wedge(n,j) \right)^2 \right)^{1/2}.$$

The Sobolev-like Hilbert space (Sobolev space) $h^{(i)}(\{A_n\};\Omega)$ is defined to be the completion

$$h^{(i)}(\{A_n\};\Omega) = \overline{\varepsilon^{(i)}(\{A_n\};\Omega)}^{\|\cdot\|_{h^{(i)}(\{A_n\};\Omega)}} .$$

As in the case of $\mathcal{H}(\{A_n\};\Omega)$, it can be shown that $\{A_n^{-1}\, y_{n,j}^{(i)} \mid n \in \mathbb{N}_{0_i}; j = 1,\ldots,2n+1\}$ is a complete orthonormal system in $h^{(i)}(\{A_n\};\Omega)$. Moreover, $h^{(i)}(\{A_n\};\Omega) \subset l^2_{(i)}(\Omega)$ (because $A_n \geq C > 0$ for all $n \in \mathbb{N}_0$). Each $f \in h^{(i)}(\{A_n\};\Omega)$ can be represented by the series expansion

$$f = \sum_{n=0_i}^{\infty} \sum_{j=1}^{2n+1} (f^{(i)})^\wedge(n,j)\, y_{n,j}^{(i)} , \tag{11}$$

where the Fourier coefficients $(f^{(i)})^\wedge(n,j)$ are given by $(f^{(i)})^\wedge(n,j) = \int_\Omega f(\xi)\cdot y_{n,j}^{(i)}(\xi)\, d\omega(\xi)$. The series expansion (11) of $f \in h^{(i)}(\{A_n\};\Omega)$ converges both with respect to $\|\cdot\|_{l^2(\Omega)}$ and $\|\cdot\|_{h^{(i)}(\{A_n\};\Omega)}$. The space $h^{(i)}(\{A_n\};\Omega)$ is a dense subspace of $l^2_{(i)}(\Omega)$.

Theorem 3.4 *Let $\{A_n\}_{n\in\mathbb{N}_0} \subset \mathbb{R}^+$ and $i \in \{1,2,3\}$. Then following statements are valid:*

(i) If $\sum_{n=0}^{\infty} \frac{(n+\frac{1}{2})}{A_n^2} < \infty$, then $h^{(i)}(\{A_n\};\Omega) \subset c_{(i)}(\Omega)$.

(ii) If $\sum_{n=0}^{\infty} \frac{(n+\frac{1}{2})^3}{A_n^2} < \infty$, then $h^{(i)}(\{A_n\};\Omega) \subset c^{(1)}_{(i)}(\Omega)$. The series expansion (11) of a vector field $f \in h^{(i)}(\{A_n\};\Omega)$ may be differentiated term by term.

(It should be noted that the conditions on $\{A_n\}_{n\in\mathbb{N}_0}$ in (i) and (ii) imply that $A_n \geq C$ for all $n \in \mathbb{N}_0$ with some constant $C > 0$.)

Proof. Statement (i) follows (as in the scalar case) by proving that the truncated Fourier series converges uniformly. This follows by aid of the Cauchy-Schwarz inequality

$$\left| \sum_{n=N}^{M} \sum_{j=1}^{2n+1} (f^{(i)})^\wedge(n,j)\, y_{n,j}^{(i)}(\xi) \right| \leq \|f\|_{h^{(i)}(\{A_n\};\Omega)} \left(\sum_{n=N}^{M} \sum_{j=1}^{2n+1} \frac{|y_{n,j}^{(i)}(\xi)|^2}{A_n^2} \right)^{1/2} , \quad M \geq N ,$$

in connection with Lemma 2.3. Statement (ii) follows from Theorem 3.2. This is obvious for $i = 1$ and will be briefly sketched for $i = 3$. The case $i = 2$ can be verified in a similar way. A function $f \in h^{(3)}(\{A_n\};\Omega)$ can be written as

$$f = \sum_{n=0_3}^{\infty} f_n ,$$

where

$$f_n := \sum_{j=1}^{2n+1} (f^{(3)})^\wedge(n,j)\, y_{n,j}^{(3)} ,$$

and

$$\|f\|^2_{h^{(3)}(\{A_n\};\Omega)} = \sum_{n=0_3}^{\infty} A_n^2 \, \|f_n\|^2_{l^2(\Omega)} .$$

It can be shown that, for $k \in \{1,2,3\}$, $f_n \cdot \varepsilon^k \in \mathrm{Harm}_n(\Omega)$ (see [10]), $f_n \cdot \varepsilon^k \in \mathcal{H}(\{A_n\};\Omega)$, and, due to Theorem 3.2, $\mathcal{H}(\{A_n\};\Omega) \subset C^{(1)}(\Omega)$. In fact,

$$\begin{aligned}
\|f \cdot \varepsilon^k\|^2_{\mathcal{H}(\{A_n\};\Omega)} &= \left\| \sum_{n=0_3}^{\infty} f_n \cdot \varepsilon^k \right\|^2_{\mathcal{H}(\{A_n\};\Omega)} \\
&= \sum_{n=0_3}^{\infty} A_n^2 \, \|f_n \cdot \varepsilon^k\|^2_{\mathcal{L}^2(\Omega)} \\
&\leq \sum_{n=0_3}^{\infty} A_n^2 \, \|f_n\|^2_{l^2(\Omega)} \\
&= \|f\|^2_{h^{(3)}(\{A_n\};\Omega)} \; < \; \infty \, .
\end{aligned}$$

This is the desired result. ■

Remark 3.5 *It is clear that Sobolev-like Hilbert spaces*

$$h(\{A_n^{(1)}\};\{A_n^{(2)}\};\{A_n^{(3)}\};\Omega) := h^{(1)}(\{A_n^{(1)}\};\Omega) \oplus h^{(2)}(\{A_n^{(2)}\};\Omega) \oplus h^{(3)}(\{A_n^{(3)}\};\Omega) \tag{12}$$

of $l^2(\Omega)$ can be defined in a canonical way, where $\{A_n^{(1)}\}_{n\in\mathbb{N}_0}$, $\{A_n^{(2)}\}_{n\in\mathbb{N}_0}$, $\{A_n^{(3)}\}_{n\in\mathbb{N}_0} \subset \mathbb{R}^+$ satisfy $A_n^{(i)} \geq C_i$ for all $n \in \mathbb{N}_0$, $i = \{1,2,3\}$ with constants $C_1, C_2, C_3 > 0$. The inner product on the space introduced in (12) is then given by

$$(f,g)_{h(\{A_n^{(1)}\};\{A_n^{(2)}\};\{A_n^{(3)}\};\Omega)} = \sum_{i=1}^{3} \sum_{n=0_i}^{\infty} \sum_{j=1}^{2n+1} (A_n^{(i)})^2 \, (f^{(i)})^\wedge(n,j) \, (g^{(i)})^\wedge(n,j) \, ,$$

and $f \in h(\{A_n^{(1)}\};\{A_n^{(2)}\};\{A_n^{(3)}\};\Omega)$ has the series expansion

$$f = \sum_{i=1}^{3} \underbrace{\left(\sum_{n=0_i}^{\infty} \sum_{j=1}^{2n+1} (f^{(i)})^\wedge(n,j) \, y_{n,j}^{(i)} \right)}_{=: \, f^{(i)}}, \tag{13}$$

where $(f^{(i)})^\wedge(n,j) = \int_\Omega f(\xi)\, y_{n,j}^{(i)}(\xi)\, d\omega(\xi)$. The series expansion (13) converges in $l^2(\Omega)$-sense, as well as in $h(\{A_n^{(1)}\};\{A_n^{(2)}\},\{A_n^{(3)}\};\Omega)$-sense. For $i \in \{1,2,3\}$ we have $f^{(i)} \in h^{(i)}(\{A_n^{(i)}\};\Omega)$. The most common case is, of course, $\{A_n^{(i)}\}_{n\in\mathbb{N}_0} = \{A_n\}_{n\in\mathbb{N}_0}$ for all $i = 1,2,3$.

3.3 Properties of the Operators $O^{(i)}$ and $o^{(i)}$

Theorem 3.6 *Let $\{A_n\}_{n\in\mathbb{N}_0} \subset \mathbb{R}^+$ satisfy $A_n \geq C$ for all $n \in \mathbb{N}_0$ with a constant $C > 0$. Then the following statements hold true:*

(i) The operator $o^{(1)}$: $\mathcal{H}(\{A_n\};\Omega) \to h^{(1)}(\{A_n\};\Omega)$ is a well-defined bijective bounded linear operator. The operator $O^{(1)}$: $h^{(1)}(\{A_n\};\Omega) \to \mathcal{H}(\{A_n\};\Omega)$ is a well-defined bijective bounded linear operator. Furthermore,

$$O^{(1)} o^{(1)} F = F \, , \qquad F \in \mathcal{H}(\{A_n\};\Omega) \, , \tag{14}$$

$$o^{(1)} O^{(1)} f = f \, , \qquad f \in h^{(1)}(\{A_n\};\Omega) \, . \tag{15}$$

(ii) For $i \in \{2,3\}$ suppose that $\{A_n\}_{n\in\mathbb{N}_0}$ satisfies additionally the inequality

$$\sum_{n=0}^{\infty} \frac{(n+\frac{1}{2})^3}{A_n^2} < \infty\,. \tag{16}$$

Then $o^{(i)} : \mathcal{H}(\{A_n\};\Omega) \to h^{(i)}(\{A_n/(n+\frac{1}{2})\};\Omega)$ is a well-defined surjective bounded linear operator. The operator $O^{(i)} : h^{(i)}(\{A_n\};\Omega) \to \mathcal{H}(\{A_n/(n+\frac{1}{2})\};\Omega)$ is a well-defined injective bounded linear operator.

Proof. Statement (i): Assume that $F \in \mathcal{H}(\{A_n\};\Omega)$. Then it can be shown that

$$\begin{aligned}
\|o^{(1)}F\|^2_{h^{(1)}(\{A_n\};\Omega)} &= \left\| \sum_{n=0}^{\infty}\sum_{j=1}^{2n+1} F^\wedge(n,j)\, y^{(1)}_{n,j} \right\|^2_{h^{(1)}(\{A_n\};\Omega)} \\
&= \sum_{n=0}^{\infty}\sum_{j=1}^{2n+1} A_n^2 \left(F^\wedge(n,j)\right)^2 \\
&= \|F\|^2_{\mathcal{H}(\{A_n\};\Omega)}\,.
\end{aligned}$$

Analogously, for $f \in h^{(1)}(\{A_n\};\Omega)$,

$$\begin{aligned}
\left\|O^{(1)}f\right\|^2_{\mathcal{H}(\{A_n\};\Omega)} &= \left\| \sum_{n=0}^{\infty}\sum_{j=1}^{2n+1} (f^{(i)})^\wedge(n,j)\, Y_{n,j} \right\|^2_{\mathcal{H}(\{A_n\};\Omega)} \\
&= \sum_{n=0}^{\infty}\sum_{j=1}^{2n+1} A_n^2 \left((f^{(i)})^\wedge(n,j)\right)^2 \\
&= \|f\|^2_{h^{(1)}(\{A_n\};\Omega)}\,.
\end{aligned}$$

Consequently, $o^{(1)}$ and $O^{(1)}$ are well-defined, injective and bounded linear operators. Due to the properties

$$\begin{aligned}
O^{(1)}o^{(1)}Y_{n,j} &= Y_{n,j}\,, \\
o^{(1)}O^{(1)}y^{(1)}_{n,j} &= y^{(1)}_{n,j}
\end{aligned}$$

for all $n \in \mathbb{N}_0$ and all $j = 1,\ldots,2n+1$, the identities (14) and (15) are verified, and $O^{(1)}$ and $o^{(1)}$ also are surjective.

Statement (ii): Suppose that $F \in \mathcal{H}(\{A_n\};\Omega)$. Then, due to Theorem 3.2, F is a member of class $\mathcal{C}^{(1)}(\Omega)$ and the Fourier series expansion of F may be differentiated term by term. Hence,

$$\begin{aligned}
\|o^{(1)}F\|^2_{h^{(i)}(\{A_n/(n+\frac{1}{2})\};\Omega)} &= \left\| \sum_{n=0_i}^{\infty}\sum_{j=1}^{2n+1} F^\wedge(n,j)\, (\mu_n^{(i)})^{1/2}\, y^{(i)}_{n,j} \right\|^2_{h^{(i)}(\{A_n/(n+\frac{1}{2})\};\Omega)} \\
&= \sum_{n=0_i}^{\infty} \frac{A_n^2}{(n+\frac{1}{2})^2}\, n(n+1) \left(F^\wedge(n,j)\right)^2 \\
&\leq 2\,\|F\|^2_{\mathcal{H}(\{A_n\};\Omega)}\,.
\end{aligned}$$

Therefore, $o^{(i)}$ is a well-defined bounded linear operator. The operator $o^{(i)}$ also is surjective, because for $f \in h^{(i)}(\{A_n/(n+\frac{1}{2})\};\Omega)$

$$F := \sum_{n=0_i}^{\infty} \sum_{j=1}^{2n+1} (f^{(i)})^\wedge(n,j)\, (\mu_n^{(i)})^{-1/2}\, Y_{n,j}$$

is a function in $\mathcal{H}(\{A_n\};\Omega)$ with $O^{(i)}F = f$.

Let now f be of class $h^{(i)}(\{A_n\};\Omega)$. According to Theorem 3.4, we have $f \in c^{(1)}(\Omega)$, and the Fourier series expansion of f can be differentiated term by term. Consequently, it follows that

$$\begin{aligned} \|O^{(i)}f\|^2_{\mathcal{H}(\{A_n/(n+\frac{1}{2})\};\Omega)} &= \left\| \sum_{n=0_i}^{\infty} \sum_{j=1}^{2n+1} (f^{(i)})^\wedge(n,j)\, (\mu_n^{(i)})^{1/2}\, Y_{n,j} \right\|^2_{\mathcal{H}(\{A_n/(n+\frac{1}{2})\};\Omega)} \\ &= \sum_{n=0_i}^{\infty} \sum_{j=1}^{2n+1} \frac{A_n^2}{(n+\frac{1}{2})^2}\, n(n+1) \left((f^{(i)})^\wedge(n,j) \right)^2 \\ &\leq 2\, \|f\|^2_{h^{(i)}(\{A_n\};\Omega)} \,. \end{aligned}$$

Hence, $O^{(i)}$ is a well-defined bounded linear operator. The injectivity follows from the fact that

$$O^{(i)} y_{n,j}^{(i)} = (\mu_n^{(i)})^{1/2}\, Y_{n,j} \neq 0 \,.$$

for all $n \in \mathbb{N}_{0_i}$; $j = 1, \ldots, 2n+1$. ■

3.4 Bounded Linear Functionals on $\mathcal{H}(\{A_n\};\Omega)$

At first, we discuss the representation of bounded linear functionals on the scalar Hilbert spaces $\mathcal{H}(\{A_n\};\Omega)$.

Theorem 3.7 *Let $\{A_n\}_{n\in\mathbb{N}_0} \subset \mathbb{R}^+$ satisfy $A_n \geq C$ for all $n \in \mathbb{N}_0$ with some constant $C > 0$. Let $\mathcal{L} : \mathcal{H}(\{A_n\};\Omega) \to \mathbb{R}$ be a bounded linear functional. Then there exists a uniquely determined representer $L \in \mathcal{H}(\{A_n\};\Omega)$ of $\mathcal{L}$, such that*

$$\mathcal{L}F = (F, L)_{\mathcal{H}(\{A_n\};\Omega)} \tag{17}$$

for all $F \in \mathcal{H}(\{A_n\};\Omega)$. This representer is given by

$$L := \sum_{n=0}^{\infty} \sum_{j=1}^{2n+1} \frac{1}{A_n^2}\, (\mathcal{L}\, Y_{n,j})\, Y_{n,j} \,. \tag{18}$$

Proof. According to the Riesz representation theorem, there exists an $L \in \mathcal{H}(\{A_n\};\Omega)$ satisfying (17). The computation of the Fourier coefficients with respect to the complete orthonormal system $\{A_n^{-1}\, Y_{n,j}\}_{n\in\mathbb{N}_0;\, j=1,\ldots,2n+1}$ shows us that

$$\left(L, A_n^{-1}\, Y_{n,j}\right)_{\mathcal{H}(\{A_n\};\Omega)} = A_n^{-1}\, \mathcal{L}Y_{n,j} \,,$$

and, consequently, the series expansion (18) is valid. ■

Our results lead us to the following theorem.

Theorem 3.8 *Let $\{A_n\}_{n\in\mathbb{N}_0} \subset \mathbb{R}^+$ satisfy condition (10). Then the Hilbert space $\mathcal{H}(\{A_n\};\Omega)$ is a reproducing kernel Hilbert space, i.e., there exists a kernel $K : \Omega\times\Omega \to \mathbb{R}$, $(\xi,\eta) \mapsto K(\xi,\eta)$, satisfying the following properties:*

(i) $K(\xi,\cdot) \in \mathcal{H}(\{A_n\};\Omega)$ for all fixed $\xi \in \Omega$.

(ii) For all $F \in \mathcal{H}(\{A_n\};\Omega)$ the property $(F, K(\xi,\cdot))_{\mathcal{H}(\{A_n\};\Omega)} = F(\xi)$ holds for all $\xi \in \Omega$.

This reproducing kernel K is uniquely determined and admits the series expansion

$$K(\xi,\eta) = \sum_{n=0}^{\infty}\sum_{j=1}^{2n+1} \frac{1}{A_n^2}\, Y_{n,j}(\xi)\, Y_{n,j}(\eta)\,, \qquad (\xi,\eta) \in \Omega\times\Omega\,.$$

Proof. This result is well known and can, for example, be found in [9], [10]. ∎

An immediate consequence of Theorem 3.7 and Theorem 3.8 is the following result.

Lemma 3.9 *Let the assumptions and the notation be the same as in Theorem 3.8. Suppose that $\mathcal{L}$ is a bounded linear functional on $\mathcal{H}(\{A_n\};\Omega)$. Then, the representer of $\mathcal{L}$ is given by*

$$L(\eta) = \mathcal{L}_\xi K(\xi,\eta)\,, \qquad \eta \in \Omega\,,$$

where K is the reproducing kernel, and the index ξ means that $\mathcal{L}$ is applied to K as a function in $\mathcal{H}(\{A_n\};\Omega)$ of the first variable. In particular, the evaluation functional

$$\mathcal{L} : \mathcal{H}(\{A_n\};\Omega) \to \mathbb{R}\,, \qquad F \mapsto F(\xi)\,,$$

in the point $\xi \in \Omega$ is bounded and has the representer

$$L(\eta) = K(\xi,\eta) = \sum_{n=0}^{\infty}\sum_{j=1}^{2n+1} \frac{1}{A_n^2}\, Y_{n,j}(\xi)\, Y_{n,j}(\eta) = \sum_{n=0}^{\infty} \frac{(2n+1)}{4\pi\, A_n^2}\, P_n(\xi\cdot\eta)\,, \qquad \eta \in \Omega\,.$$

3.5 Bounded Linear Functionals on $h^{(i)}(\{A_n\};\Omega)$

Now we come to the vectorial case. At first, the representation of bounded linear functionals on $h^{(i)}(\{A_n\};\Omega)$ is discussed in general. After these considerations, some relevant examples of bounded linear functionals are given.

Theorem 3.10 *For $i \in \{1,2,3\}$ let $\{A_n\}_{n\in\mathbb{N}_0} \subset \mathbb{R}^+$ satisfy $A_n \geq C$ for all $n \in \mathbb{N}_0$ with some constant $C > 0$. Assume that $\mathcal{L} : h^{(i)}(\{A_n\};\Omega) \to \mathbb{R}$ is a bounded linear functional. Then there exists a uniquely determined representer $l \in h^{(i)}(\{A_n\};\Omega)$ of $\mathcal{L}$, i.e.,*

$$\mathcal{L}f = (f, l)_{h^{(i)}(\{A_n\};\Omega)}$$

for all $f \in h^{(i)}(\{A_n\};\Omega)$. This representer l is given by

$$l := \sum_{n=0_i}^{\infty}\sum_{j=1}^{2n+1} \frac{1}{A_n^2}\, (\mathcal{L}y_{n,j}^{(i)})\, y_{n,j}^{(i)}\,. \tag{19}$$

Proof. The existence of a uniquely determined $l \in h^{(i)}(\{A_n\};\Omega)$ follows from the Riesz representation theorem. In order to verify (19), we compute the Fourier coefficients of l with respect to the complete orthonormal system $\{A_n^{-1}\, y_{n,j}^{(i)}\}_{n\in\mathbb{N}_{0_i};\, j=1,\ldots,2n+1}$. Obviously,

$$\left(l, A_n^{-1}\, y_{n,j}^{(i)}\right)_{h^{(i)}(\{A_n\};\Omega)} = A_n^{-1}\, \mathcal{L} y_{n,j}^{(i)} .$$

This concludes the proof. ■

Lemma 3.11 *Assume that $i \in \{2,3\}$, and suppose that $\{A_n\}_{n\in\mathbb{N}_0} \subset \mathbb{R}^+$ satisfies (16). Then $\mathcal{L} : h^{(i)}(\{A_n\};\Omega) \to \mathbb{R}$, $f \mapsto \mathcal{L}f := O_\xi^{(i)} f(\xi)$, for some fixed $\xi \in \Omega$, is a bounded linear functional with the representer*

$$l := \sum_{n=0_i}^{\infty} \sum_{j=1}^{2n+1} \frac{1}{A_n^2}\, (n(n+1))^{1/2}\, Y_{n,j}(\xi)\, y_{n,j}^{(i)} . \tag{20}$$

Proof. That $\mathcal{L}$ is well-defined is clear because $h^{(i)}(\{A_n\};\Omega)$ is a subspace of $c_{(i)}^{(1)}(\Omega)$, according to Theorem 3.4. In order to show the boundedness of $\mathcal{L}$, note that $(n(n{+}1))^{1/2}\, Y_{n,j}(\xi) = O_\xi^{(i)} y_{n,j}^{(i)}(\xi)$. As the series expansion of $f \in h^{(i)}(\{A_n\};\Omega)$ may be differentiated term by term,

$$O_\xi^{(i)} f(\xi) = \sum_{n=0_i}^{\infty} \sum_{j=1}^{2n+1} (f^{(i)})^\wedge(n,j)\, O_\xi^{(i)} y_{n,j}^{(i)}(\xi) ,$$

and it is obvious that $\mathcal{L}f = (f,l)_{h^{(i)}(\{A_n\};\Omega)}$ with l given by (20) if $l \in h^{(i)}(\{A_n\};\Omega)$. This implies

$$|\mathcal{L}f| = \left|(f,l)_{h^{(i)}(\{A_n\};\Omega)}\right| \leq \|l\|_{h^{(i)}(\{A_n\};\Omega)}\, \|f\|_{h^{(i)}(\{A_n\};\Omega)} ,$$

and $\mathcal{L}$ is bounded. In order to prove $l \in h^{(i)}(\{A_n\};\Omega)$, we estimate its norm as follows:

$$\begin{aligned} \|l\|^2_{h^{(i)}(\{A_n\};\Omega)} &= \sum_{n=0_i}^{\infty} \sum_{j=1}^{2n+1} A_n^2\, \frac{n(n+1)}{A_n^4}\, (Y_{n,j}(\xi))^2 \\ &= \sum_{n=0_i}^{\infty} \frac{n(n+1)}{A_n^2}\, \frac{(2n+1)}{4\pi} \;<\; \infty . \end{aligned}$$

This shows the wanted result. ■

Finally, we are concerned with the following result.

Lemma 3.12 *For $i \in \{1,2,3\}$ assume that $\{A_n\}_{n\in\mathbb{N}_0} \subset \mathbb{R}^+$ satisfies condition (10). Let $a \in \mathbb{R}^3$, $a \neq 0$, and let $\xi \in \Omega$ be a fixed point on Ω. Then $\mathcal{L} : h^{(i)}(\{A_n\};\Omega) \to \mathbb{R}$, $f \mapsto \mathcal{L}f := a \cdot f(\xi)$, is a bounded linear functional possessing the uniquely determined representer $l \in h^{(i)}(\{A_n\};\Omega)$, given by*

$$l := \sum_{n=0_i}^{\infty} \sum_{j=1}^{2n+1} \frac{1}{A_n^2} \left(a \cdot y_{n,j}^{(i)}(\xi)\right) y_{n,j}^{(i)} . \tag{21}$$

Proof. First we check that l, given by (21), is in $h^{(i)}(\{A_n\};\Omega)$: According to Lemma 2.3

$$\begin{aligned}\|l\|^2_{h^{(i)}(\{A_n\};\Omega)} &= \sum_{n=0_i}^{\infty}\sum_{j=1}^{2n+1}\frac{|a\cdot y^{(i)}_{n,j}(\xi)|^2}{A_n^2}\\ &\leq \sum_{n=0_i}^{\infty}\sum_{j=1}^{2n+1}\frac{|a|^2|y^{(i)}_{n,j}(\xi)|^2}{A_n^2}\\ &\leq |a|^2\cdot\left(\sum_{n=0_i}^{\infty}\frac{(2n+1)}{4\pi A_n^2}\right) < \infty\,.\end{aligned}$$

Now we have

$$\mathcal{L}f = \sum_{n=0_i}^{\infty}\sum_{j=1}^{2n+1}(f^{(i)})^\wedge(n,j)\left(a\cdot y^{(i)}_{n,j}(\xi)\right) = (f,l)_{h^{(i)}(\{A_n\};\Omega)}$$

This proves Lemma 3.12. ∎

Finally, we remark that the representers l, given by (20) or (21), of the bounded linear functionals $\mathcal{L}$ discussed in Lemma 3.11 and Lemma 3.12, respectively, are for certain choices of $\{A_n\}_{n\in\mathbb{N}_0}$ available as elementary functions. Such choices of $\{A_n\}_{n\in\mathbb{N}_0}$ will be discussed later on.

4 Splines

Before splines are introduced, we give a brief motivation explaining why spline interpolation or spline smoothing in weighted Sobolev spaces $\mathcal{H}(\{A_n\};\Omega)$ or $h^{(i)}(\{A_n\};\Omega)$ are appropriate for the modelling of the geostrophic wind field. After the definition of scalar and vectorial splines, three spline interpolation problems (and the corresponding spline smoothing problems) are presented and their properties are discussed. Finally, some special types of Sobolev spaces $\mathcal{H}(\{A_n\};\Omega)$ and $h^{(3)}(\{A_n\};\Omega)$ are introduced in which the spline functions are available as elementary functions.

4.1 An Extension of Helly's Theorem

It is sensible to assume that the air pressure P and the geostrophic wind $v_{\tan}$ are continuous, such that P is a function of class $\mathcal{C}^{(1)}(\Omega)$ and that $L^*P = B\,\rho\,v_{\tan}$ is a function in $c(\Omega)$. As mentioned in our introduction, we want either to reconstruct $\xi\mapsto P(\xi)$, $\xi\in\Omega$, from the given data

$$\left\{\left(\xi_i^N, P(\xi_i^N)+\epsilon_i\right)\middle|\; i=1,\ldots,N\right\} \tag{22}$$

or to approximate $\xi\mapsto L^*_\xi P(\xi) = B(\xi)\,\rho(\xi)\,v_{\tan}(\xi)$, $\xi\in\Omega$, with the help of the given data

$$\left\{\left(\xi_i^N, B(\xi_i^N)\,\rho(\xi_i^N)\,(v_{\tan}(\xi_i^N)+\epsilon_i)\right)\middle|\; i=1,\ldots,N\right\}, \tag{23}$$

where $\epsilon_1,\ldots,\epsilon_N$ are scalar and vectorial measurement errors, respectively. As the spaces $\mathcal{H}(\{A_n\};\Omega)$ and $h^{(3)}(\{A_n\};\Omega)$ have a very convenient mathematical structure, we would like to approximate P and $L^*P = B\,\rho\,v_{\tan}$ in one of these spaces with the help of the data (22) and (23), respectively.

The next theorem explains why this approach is suitable.

Theorem 4.1 *Let $\mathcal{M}$ be a dense and convex subset in a normed linear (not necessarily complete) space $\mathcal{X}$ with norm $\|\cdot\|_{\mathcal{X}}$, and let $\mathcal{L}_1, \ldots, \mathcal{L}_N$ be N bounded linear functionals on $\mathcal{X}$. Suppose that F is of class $\mathcal{X}$. Given $\varepsilon > 0$, there exists an element $G \in \mathcal{M}$ with the following properties:*

(i) $\|F - G\|_{\mathcal{X}} \leq \varepsilon$,

(ii) $\mathcal{L}_i F = \mathcal{L}_i G$ *for* $i = 1, \ldots, N$.

The proof is an extension of Helly's theorem that can be founded in [28].

In order to apply Theorem 4.1 to our particular problems, we have to check that all assumptions of Theorem 4.1 are satisfied:

If the sequence $\{A_n\}_{n\in\mathbb{N}_0} \subset \mathbb{R}^+$ satisfies (10), $\mathcal{H}(\{A_n\};\Omega)$ is a subspace of the complete normed vector space $(\mathcal{C}(\Omega); \|\cdot\|_{\mathcal{C}(\Omega)})$, due to Theorem 3.2. Obviously, $\mathcal{H}(\{A_n\};\Omega)$ is convex in $\mathcal{C}(\Omega)$. It is well-known that $\overline{\{Y_{n,j} \,|\, n \in \mathbb{N}_0; j = 1, \ldots, 2n+1\}}^{\|\cdot\|_{\mathcal{C}(\Omega)}} = \mathcal{C}(\Omega)$. As $\mathcal{H}(\{A_n\};\Omega)$ contains all spherical harmonics $Y_{n,j}$ it is also dense in $\mathcal{C}(\Omega)$. Hence, it remains to verify that evaluation functionals, which represent the air pressure data (22), are bounded in $(\mathcal{C}(\Omega); \|\cdot\|_{\mathcal{C}(\Omega)})$: Obviously, $\left|F(\xi_i^N)\right| \leq \|F\|_{\mathcal{C}(\Omega)}$ for all $F \in \mathcal{C}(\Omega)$. Consequently, Theorem 4.1 implies that for every function $F \in \mathcal{C}(\Omega)$, there exists a function $G \in \mathcal{H}(\{A_n\};\Omega)$ such that

(i) $\sup_{\xi\in\Omega} |G(\xi) - F(\xi)| \leq \varepsilon$,

(ii) $G(\xi_i^N) = F(\xi_i^N)$ for $i = 1, \ldots, N$.

$(c_{(3)}(\Omega); \|\cdot\|_{c(\Omega)})$ is a complete normed space. Moreover, due to Theorem 3.4, $h^{(3)}(\{A_n\};\Omega)$ is a subspace of $c_{(3)}(\Omega)$ provided that (10) holds true. As a subspace of $c_{(3)}(\Omega)$, the space $h^{(3)}(\{A_n\};\Omega)$ is obviously convex. Furthermore, $h^{(3)}(\{A_n\};\Omega)$ is dense in $c_{(3)}(\Omega)$, because it contains all vector spherical harmonics of type 3. It remains to prove that the linear functionals representing our (exact) data are bounded. Vectorial data $(\xi_i^N, f(\xi_i^N))$ (where in our case $f(\xi_i^N) := B(\xi_i^N)\,\rho(\xi_i^N)\,v_{\tan}(\xi_i^N))$, $i = 1, \ldots, N$, lead to scalar data of the following type

$$\mathcal{L}_i^a : c_{(3)}(\Omega) \to \mathbb{R}, \quad g \mapsto a \cdot g(\xi_i^N), \tag{24}$$

where $a \in \mathbb{R}^3$, $|a| = 1$, is a fixed vector in $\mathbb{R}^3$. The choices of interest for a are either the canonical unit vectors $\varepsilon^1, \varepsilon^2, \varepsilon^3$ of $\mathbb{R}^3$ or a basis of the tangential space in $\xi_i^N \in \Omega$. In order to apply Theorem 4.1, it remains to show that $\mathcal{L}_i^a$, given by (24), is bounded: $|\mathcal{L}_i^a g| = |a \cdot g(\xi_i^N)| \leq |a|\,|g(\xi_i^N)| \leq |a|\,\|g\|_{c(\Omega)}$ for all $g \in c(\Omega)$. Suppose now that (10) is valid, and let $\varepsilon > 0$ be given. Then Theorem 4.1 tells us that there exists a function $g \in h^{(3)}(\{A_n\};\Omega)$ such that

(i) $\sup_{\xi\in\Omega} |g(\xi) - B(\xi)\,\rho(\xi)\,v_{\tan}(\xi)| \leq \varepsilon$,

(ii) $a \cdot g(\xi_i^N) = a \cdot \left(B(\xi_i^N)\,\rho(\xi_i^N)\,v_{\tan}(\xi_i^N)\right)$ for $i = 1, \ldots, N$ and for a finite number of vectors $a \in \mathbb{R}^3$ with $|a| = 1$.

This motivates why it makes sense to compute in $h^{(3)}(\{A_n\};\Omega)$ an approximating spline from the (measured) data (23) and to expect that this spline is a good approximation of the field

$\xi \mapsto B(\xi)\,\rho(\xi)\,v_{\tan}(\xi)$, $\xi \in \Omega$, in $c_{(3)}(\Omega)$. Analogously, it is reasonable to compute an approximating spline in $\mathcal{H}(\{A_n\};\Omega)$ as an approximation of the air pressure P from the measured data (22). However, it should be kept in mind that Theorem 4.1 merely is an existence theorem. It gives no ideas, how to find a $G \in \mathcal{H}(\{A_n\};\Omega)$ or a $g \in h^{(3)}(\{A_n\};\Omega)$ satisfying the conditions (i) and (ii).

4.2 Geostrophic Wind Field Modelling by Spherical Splines

We begin with the wind field modelling from discrete pressure data (PDP) as given in (22).

4.2.1 Scalar Splines

Scalar spline theory, as introduced here, has been known since the early eighties (see [8]) and has been applied numerically with success to a large number of problems (see, for example, [10]).

Definition 4.2 *Let $\{A_n\}_{n\in\mathbb{N}_0} \subset \mathbb{R}^+$ satisfy $A_n \geq C$ for all $n \in \mathbb{N}_0$ with some constant $C > 0$. Let $\mathcal{L}_1,\ldots,\mathcal{L}_N$ be N (linearly independent) bounded linear functionals $\mathcal{L}_k : \mathcal{H}(\{A_n\};\Omega) \to \mathbb{R}$, $F \mapsto \mathcal{L}_k F$, with representers $L_1,\ldots,L_N$. Then every function of the form*

$$S = \sum_{k=1}^{N} \alpha_k\, L_k$$

(in $\mathcal{H}(\{A_n\};\Omega)$) with coefficients $\alpha_1,\ldots,\alpha_N \in \mathbb{R}$ is called an $\mathcal{H}(\{A_n\};\Omega)$-spline relative to $\mathcal{L}_1,\ldots,\mathcal{L}_N$. The space of all $\mathcal{H}(\{A_n\};\Omega)$-splines relative to $\mathcal{L}_1,\ldots,\mathcal{L}_N$ is denoted by

$$\mathrm{Spline}_{\mathcal{H}(\{A_n\};\Omega)}(\mathcal{L}_1,\ldots,\mathcal{L}_N) = \mathrm{span}\,\{L_k \,|\, k = 1,\ldots,N\}\,.$$

Next, we formulate the spline interpolation problem providing the desired approximation of the air pressure data.

Problem 4.3 *Suppose that $\{A_n\}_{n\in\mathbb{N}_0} \subset \mathbb{R}^+$ satisfies condition (16). Let $\{\eta_1^M,\ldots,\eta_M^M\} \subset \Omega$ be a set of M points such that*

$$\mathcal{L}_k : \mathcal{H}(\{A_n\};\Omega) \to \mathbb{R}\,, \quad F \mapsto \mathcal{L}_k F := F(\eta_k^M)\,,$$

$k = 1,\ldots,M$, are M (linearly independent) bounded linear functionals. Denote by L_k the representer of $\mathcal{L}_k$, $k \in \{1,\ldots,M\}$. The spline interpolation problem for the determination of the air pressure reads as follows:

Find a spline $S = \sum_{k=1}^{M} \alpha_k\, L_k \in \mathrm{Spline}_{\mathcal{H}(\{A_n\};\Omega)}(\mathcal{L}_1,\ldots,\mathcal{L}_M)$ such that

$$S(\xi_j^N) = \sum_{k=1}^{M} \alpha_k\, L_k(\xi_j^N) \overset{!}{=} P(\xi_j^N)\,, \qquad j = 1,\ldots,N\,. \tag{25}$$

The geostrophic wind $v_{\tan}$ can then be reconstructed via the formula

$$B(\xi)\,\rho(\xi)\,v_{\tan}(\xi) \approx L_\xi^* S(\xi)\,, \qquad \xi \in \Omega\,.$$

(Note, that L^*S is well-defined because $\mathcal{H}(\{A_n\};\Omega)$ is a subspace of $\mathcal{C}^{(1)}(\Omega)$, due to Theorem 3.2.)

Clearly, Problem 4.3 is always solvable, provided that dim $\mathrm{Spline}_{\mathcal{H}(\{A_n\};\Omega)}(\mathcal{L}_1,\ldots,\mathcal{L}_M) \geq N$. We distinguish from now on in Problem 4.3 two cases: (i) the set $\{\eta_1^M,\ldots,\eta_M^M\}$ coincides with the set of measurement points $\{\xi_1^N,\ldots,\xi_N^N\}$, and (ii) $\{\eta_1^M,\ldots,\eta_M^M\}$ does not coincide with $\{\xi_1^N,\ldots,\xi_N^N\}$. Furthermore, we denote the linear system in (25) by $\mathbf{A}\,x = b$.

Case (ii): If $\mathrm{rank}(\mathbf{A}) = M$ and if we are given exact (non-noisy) data, then we solve the normal equations

$$\mathbf{A}^T\mathbf{A}\,x = \mathbf{A}^T b\,.$$

In all other cases, we solve the equations

$$\mathbf{A}^T\mathbf{A}\,x + \lambda\,\mathbf{Id}\,x = \mathbf{A}^T b\,, \tag{26}$$

where $\lambda > 0$ is a suitable constant, the so-called smoothing parameter, and where $\mathbf{Id}$ is the identity matrix. The linear system (26) can be interpreted as a Tikhonov regularization.

Case (i): This is mathematically the more interesting case, because $L_k(\xi_j^N) = (L_j, L_k)_{\mathcal{H}(\{A_n\};\Omega)}$. The matrix $\mathbf{A}$ of the linear equation system (25) is then given by

$$\mathbf{A} = \Big((L_j, L_k)_{\mathcal{H}(\{A_n\};\Omega)} \Big)_{j,k=1,\ldots,N}\,.$$

If rank $(\mathbf{A}) = N$, i.e., $\mathcal{L}_1,\ldots,\mathcal{L}_N$ are linearly independent, and if we are given exact data, then we solve

$$\mathbf{A}\,x = b\,. \tag{27}$$

In case of noisy data or linear dependence of $\mathcal{L}_1,\ldots,\mathcal{L}_N$, we solve

$$\mathbf{A}\,x + \lambda\,\mathbf{Id}\,x = b, \tag{28}$$

where $\lambda > 0$ is a smoothing parameter which weights between fitting the data and smoothness of the solution. (The smaller λ, the closer we come to the interpolation scenario.) Both cases (27) and (28) are described by the following theorem in the context of scalar spline theory.

Theorem 4.4 *Let $\{A_n\}_{n\in\mathbb{N}_0} \subset \mathbb{R}^+$ satisfy $A_n \geq C$ for all $n \in \mathbb{N}_0$ with some constant $C > 0$. Suppose $\mathcal{L}_k : \mathcal{H}(\{A_n\};\Omega) \to \mathbb{R}$, $F \mapsto \mathcal{L}_k F$, $k = 1,\ldots,N$, are N bounded linear functionals, and denote the representer of $\mathcal{L}_k$ by L_k. Let $\lambda \geq 0$ be a non-negative real number, and let $b_1,\ldots,b_N \in \mathbb{R}$ be given values. If $\lambda > 0$, there exists one and only one spline $S = \sum_{k=1}^N \alpha_k L_k$ in $\mathrm{Spline}_{\mathcal{H}(\{A_n\};\Omega)}(\mathcal{L}_1,\ldots,\mathcal{L}_N)$ with coefficient vector $\alpha = (\alpha_1,\ldots,\alpha_N)^T$ which minimizes the functional*

$$\mu_\lambda(\alpha) := \sum_{k=1}^N (\mathcal{L}_k S - b_k)^2 + \lambda\,\|S\|^2_{\mathcal{H}(\{A_n\};\Omega)}\,.$$

In case $\mathcal{L}_1,\ldots,\mathcal{L}_N$ are linearly independent, there exists also a uniquely determined spline S which minimizes μ_λ if $\lambda = 0$. In both cases, the coefficient vector $\alpha = (\alpha_1,\ldots,\alpha_N)^T$ of S is the uniquely determined solution of the linear system

$$\left(\sum_{k=1}^N \alpha_k\,(L_k, L_j)_{\mathcal{H}(\{A_n\};\Omega)}\right) + \lambda\,\alpha_j = b_j\,, \qquad j = 1,\ldots,N\,.$$

Proof. The proof can be found in, for example, [10]. ∎

A crucial point is, of course, the choice of the smoothing parameter. This will be discussed later, after we will have given an analogous result for the case of vectorial splines.

4.2.2 Vectorial Splines

Definition 4.5 *For $i \in \{1,2,3\}$ let $\{A_n\}_{n\in\mathbb{N}_0} \subset \mathbb{R}^+$ satisfy $A_n \geq C$ for all $n \in \mathbb{N}_0$ with some constant $C > 0$. Let $\mathcal{L}_1, \ldots, \mathcal{L}_N$ be N (linearly independent) bounded linear functionals $\mathcal{L}_k : h^{(i)}(\{A_n\};\Omega) \to \mathbb{R}$, $f \mapsto \mathcal{L}_k f$, with representers $l_1, \ldots, l_N$. Then every function*

$$S = \sum_{k=1}^{N} \alpha_k \, l_k$$

(in $h^{(i)}(\{A_n\};\Omega)$) with coefficients $\alpha_1, \ldots, \alpha_N \in \mathbb{R}$ is called an $h^{(i)}(\{A_n\};\Omega)$-spline relative to $\mathcal{L}_1, \ldots, \mathcal{L}_N$. The space of all $h^{(i)}(\{A_n\};\Omega)$-splines relative to $\mathcal{L}_1, \ldots, \mathcal{L}_N$ is denoted by

$$\mathrm{Spline}_{h^{(i)}(\{A_n\};\Omega)}(\mathcal{L}_1, \ldots, \mathcal{L}_N) = \mathrm{span}\{l_k \mid k = 1, \ldots, N\} \ .$$

Obviously, Definition 4.5 is analogous to the introduction of spherical splines in Definition 4.2. If $\mathcal{L}_1, \ldots, \mathcal{L}_n$ are linearly independent then it follows that $\dim \mathrm{Spline}_{h^{(i)}(\{A_n\};\Omega)}(\mathcal{L}_1, \ldots, \mathcal{L}_N) = N$. It is clear that splines can analogously be defined for a space $h(\{A_n^{(1)}\};\{A_n^{(2)}\};\{A_n^{(3)}\};\Omega)$, but this will not be discussed further in this publication.

From now on, we will restrict our attention to the case $i = 3$, and we will formulate spline (interpolation) problems that are appropriate for wind field modelling from vectorial data as given in (23). We remind the reader of Lemma 3.11 and Lemma 3.12, in which the boundedness and representation of certain types of linear functionals have been investigated.

Problem 4.6 *Suppose $\{A_n\}_{n\in\mathbb{N}_0} \subset \mathbb{R}^+$ satisfies condition (16) and let $\{\eta_1^M, \ldots, \eta_M^M\} \subset \Omega$ be a set of M points such that*

$$\mathcal{L}_k : h^{(3)}(\{A_n\};\Omega) \to \mathbb{R}\,, \quad f \mapsto \mathcal{L}_k f := O^{(3)}_{\eta_k^M} f(\eta_k^M)\,, \qquad k = 1, \ldots, M\,,$$

are M (linearly independent) bounded linear functionals. Denote by $l_k \in h^{(3)}(\{A_n\};\Omega)$ the representer of $\mathcal{L}_k$, $k \in \{1, \ldots, M\}$. This leads to the following two interpolation problems:

(a) Find a spline $S = \sum\limits_{k=1}^{M} \alpha_k \, l_k \in$ $\mathrm{Spline}_{h^{(3)}(\{A_n\};\Omega)}(\mathcal{L}_1, \ldots, \mathcal{L}_M)$ such that

$$\varepsilon^i \cdot S(\xi_j^N) = \sum_{k=1}^{M} \alpha_k \, \left(\varepsilon^i \cdot l_k(\xi_j^N)\right) \stackrel{!}{=} B(\xi_j^N)\,\rho(\xi_j^N)\,\left(\varepsilon^i \cdot v_{\tan}(\xi_j^N)\right) \tag{29}$$

for $j = 1, \ldots, N$ and $i = 1, 2, 3$.

(b) Find a spline $S = \sum\limits_{k=1}^{M} \alpha_k \, l_k \in$ $\mathrm{Spline}_{h^{(3)}(\{A_n\};\Omega)}(\mathcal{L}_1, \ldots, \mathcal{L}_M)$ such that

$$a_j^i \cdot S(\xi_j^N) = \sum_{k=1}^{M} \alpha_k \left(a_j^i \cdot l_k(\xi_j^N)\right) \stackrel{!}{=} B(\xi_j^N)\,\rho(\xi_j^N)\,\left(a_j^i \cdot v_{\tan}(\xi_j^N)\right) \tag{30}$$

for $j = 1, \ldots, N$, and for $i = 1, 2$, where $\{a_j^1, a_j^2\}$ is an orthonormal basis of the tangential space to Ω in the point ξ_j^N. If we use polar coordinates, we can (apart from the poles) choose $a_j^1 := \varepsilon^\varphi$, $a_j^2 := \varepsilon^t$.

It should be pointed out that it seems not possible to predict in Problem 4.6 (a) or Problem 4.6 (b), whether the matrix of the interpolation problem has maximal rank, i.e., in case (a)

$$\operatorname{rank}\left(\left(\varepsilon^i \cdot l_k(\xi_j^N)\right)_{\substack{j=1,\dots,N;\, i=1,2,3;\\ k=1,\dots,M}}\right) = \min\{3N, M\}$$

and in case (b)

$$\operatorname{rank}\left(\left(a_j^i \cdot l_k(\xi_j^N)\right)_{\substack{j=1,\dots,N;\, i=1,2;\\ k=1,\dots,M}}\right) = \min\{2N, M\}\,.$$

In order to compute the spline $S = \sum_{k=1}^{M} \alpha_k\, l_k$ from the given data, we have to solve a linear system

$$\mathbf{A}\, x = b\,, \tag{31}$$

where in case (a) $\mathbf{A}$ is an $(3N \times M)$-matrix and $b \in \mathbb{R}^{3N}$ and where in case (b) $\mathbf{A}$ is an $(2N \times M)$-matrix and be $b \in \mathbb{R}^{2N}$.

In order to be able to solve (31) efficiently, we want to 'transform' (31) into a linear system with a positive definite symmetric matrix. Therefore, we distinguish two cases:

Case (i): If the given data is exact (not noisy) and if in case (a) $M \leq 2N$, in case (b) $M \leq 3M$ and if in both cases $\operatorname{rank}(\mathbf{A}) = M$, then we solve the normal equations

$$\mathbf{A}^T \mathbf{A}\, x = \mathbf{A}^T b\,, \tag{32}$$

where $\mathbf{A}^T\mathbf{A}$ is now a positive definite symmetric matrix of $\operatorname{rank}(\mathbf{A}^T\mathbf{A}) = M$.

Case (ii): If the data is noisy or if any of the other conditions mentioned in (1) is violated we solve the equations

$$\mathbf{A}^T \mathbf{A}\, x + \lambda\, \mathbf{Id}\, x = \mathbf{A}^T b\,, \tag{33}$$

where $\lambda > 0$ is a suitable constant, the so-called smoothing parameter. The matrix in (33) is $\mathbf{A}^T\mathbf{A} + \lambda\,\mathbf{Id}$, which is clearly a positive definite symmetric matrix. If all assumptions of (i) are satisfied but the matrix $\mathbf{A}^T\mathbf{A}$ in (32) is extremely bad conditioned, we also go over to case (ii) and solve (33) with a very small smoothing parameter λ. (The matrix $\mathbf{A}^T\mathbf{A} + \lambda\,\mathbf{Id}$ is better conditioned than $\mathbf{A}^T\mathbf{A}$.) It is well-known from the theory of Tikhonov regularization that the uniquely determined solution of (33) is the uniquely determined element $x \in \mathbb{R}^M$ which minimizes the linear functional

$$\mu_\lambda(x) := |\mathbf{A}\, x - b|^2 + \lambda\, |x|^2\,. \tag{34}$$

The larger the smoothing parameter λ, the more weight is put on the smoothness of the solution and the fewer weight on a small discrepancy between the data b and its reconstruction $\mathbf{A}\, x$. The smoothing parameter has to be chosen in dependence of the measurement noise (and the 'numerical' noise).

One method for the determination of the parameter λ turns out to be the L-curve method: To be more precise, compute solutions x of (33) for various smoothing parameters and plot the two terms in the functional (34) in double-logarithmic scale in dependence of λ. The curve

$$\lambda \mapsto \left(\log|\mathbf{A}\, x - b|^2\,, \log|x|^2\right)$$

can be expected to be L-shaped, and theoretical considerations suggest that a good parameter λ corresponds to the corner point of the 'L'. (For more information about Tikhonov regularization and the L-curve method the reader is referred to [6] and the references mentioned therein.)

Problem 4.7 *Suppose* $\{A_n\}_{n\in\mathbb{N}_0} \subset \mathbb{R}^+$ *satisfies the condition (10).*

(a) Let $\mathcal{L}_1, \ldots, \mathcal{L}_{3N}$ *be the bounded linear functionals*

$$\mathcal{L}_{3(m-1)+i} : h^{(3)}(\{A_n\};\Omega) \to \mathbb{R}\,, \quad f \mapsto \varepsilon^i \cdot f(\xi_m^N)\,,$$

$m = 1, \ldots, N$; $i = 1, 2, 3$, *where* $\xi_1^N, \ldots, \xi_N^N$ *are the measurement points. Denote the representer of the functional* $\mathcal{L}_k$ *by* l_k, $k = 1, \ldots, 3N$. *Find a spline* $S = \sum_{k=1}^{3N} \alpha_k\, l_k$ *of class* $\mathrm{Spline}_{h^{(3)}(\{A_n\};\Omega)}(\mathcal{L}_1, \ldots, \mathcal{L}_{3N})$ *such that*

$$\begin{aligned} \varepsilon^j \cdot S(\xi_m^N) + \lambda\, \alpha_{3(m-1)+j} &= \left(\sum_{k=1}^{3N} \alpha_k\, \left(\varepsilon^j \cdot l_k(\xi_m^N)\right)\right) + \lambda\, \alpha_{3(m-1)+j} \\ &\stackrel{!}{=} B(\xi_m^N)\, \rho(\xi_m^N)\, \left(\varepsilon^j \cdot \left(v_{\tan}(\xi_m^N) + \epsilon_m\right)\right) \end{aligned} \tag{35}$$

for $m = 1, \ldots, N$; $j = 1, 2, 3$, *where* $\lambda \geq 0$.

(b) Let $\{a_m^1, a_m^2\}$ *be an orthonormal basis of the tangential space of* Ω *in the measurement point* $\xi_m^N \in \Omega$, $m \in \{1, \ldots, N\}$. *Let* $\mathcal{L}_{2(m-1)+i}$, $m \in \{1, \ldots, N\}$, $i \in \{1, 2\}$, *be the bounded linear functional*

$$\mathcal{L}_{2(m-1)+i} : h^{(3)}(\{A_n\};\Omega) \to \mathbb{R}\,, \quad f \mapsto a_m^i \cdot f(\xi_m^N)\,,$$

and denote its representer by $l_{2(m-1)+i}$. *Find a spline* $S = \sum_{k=1}^{2N} \alpha_k\, l_k$ *in the space* $\mathrm{Spline}_{h^{(3)}(\{A_n\};\Omega)}(\mathcal{L}_1, \ldots, \mathcal{L}_{2N})$ *which satisfies*

$$\begin{aligned} a_m^j \cdot S(\xi_m^N) + \lambda\, \alpha_{2(m-1)+j} &= \left(\sum_{k=1}^{2N} \alpha_k\, \left(a_m^j \cdot l_k(\xi_m^N)\right)\right) + \lambda\, \alpha_{2(m-1)+j} \\ &= B(\xi_m^N)\, \rho(\xi_m^N)\, \left(a_m^j \cdot \left(v_{\tan}(\xi_m^N) + \epsilon_m\right)\right) \end{aligned} \tag{36}$$

for $m = 1, \ldots, N$; $j = 1, 2$, *where* $\lambda \geq 0$.

In order to analyse Problem 4.7 (a) and (b), we rewrite (35) and (36) in the following way: (35) is equivalent to

$$\begin{aligned} \left(S, l_{3(m-1)+j}\right)_{h^{(3)}(\{A_n\};\Omega)} + \lambda\, \alpha_{3(m-1)+j} &= \left(\sum_{k=1}^{3N} \alpha_k\, \left(l_k, l_{3(m-1)+j}\right)_{h^{(3)}(\{A_n\};\Omega)}\right) + \lambda\, \alpha_{3(m-1)+j} \\ &\stackrel{!}{=} B(\xi_m^N)\, \rho(\xi_m^N)\, \left(\varepsilon^j \cdot \left(v_{\tan}(\xi_m^N) + \epsilon_m\right)\right) \end{aligned} \tag{37}$$

for $m = 1, \ldots, N$; $j = 1, 2, 3$, and (36) is equivalent to

$$\begin{aligned} \left(S, l_{2(m-1)+j}\right)_{h^{(3)}(\{A_n\};\Omega)} + \lambda\, \alpha_{2(m-1)+j} &= \left(\sum_{k=1}^{2N} \alpha_k (l_k, l_{2(m-1)+j})_{h^{(3)}(\{A_n\};\Omega)}\right) + \lambda\, \alpha_{2(m-1)+j} \\ &\stackrel{!}{=} B(\xi_m^N)\, \rho(\xi_m^N)\, \left(a_m^j \cdot \left(v_{\tan}(\xi_m^N) + \epsilon_m\right)\right) \end{aligned} \tag{38}$$

for $m = 1, \ldots, N$; $j = 1, 2$. The matrix of (37) or (38) is given by

$$\Big((l_k, l_m)_{h^{(3)}(\{A_n\};\Omega)}\Big)_{k,m=1,\ldots,M} + \lambda\, \mathbf{Id}\,, \tag{39}$$

where $M = 3N$ in case of (37) and $M = 2N$ in case of (38). The matrix (39) is symmetric. It is, in addition, positive definite if $\lambda > 0$ or if $l_1, \dots l_M$ are linearly independent. The parameter $\lambda > 0$ is a smoothing parameter, and for $\lambda = 0$ we get the classical interpolation case. As in the case of scalar spherical splines, the following theorem about spline interpolation and spline smoothing is valid:

Theorem 4.8 *Let $\{A_n\}_{n\in\mathbb{N}_0} \subset \mathbb{R}^+$ satisfy $A_n \geq C$ for all $n \in \mathbb{N}_0$ with some constant $C > 0$, and let $i \in \{1,2,3\}$. Let $\mathcal{L}_k : h^{(i)}(\{A_n\};\Omega) \to \mathbb{R}$, $f \mapsto \mathcal{L}_k f$, $k \in \{1,\dots,M\}$, be M bounded linear functionals, and denote the representer of $\mathcal{L}_k$ by $l_k \in h^{(i)}(\{A_n\};\Omega)$. Let $\lambda \geq 0$ be a non-negative real number and suppose $b_1,\dots,b_M \in \mathbb{R}$ are given values. (Usually the b_k are somehow related to the $\mathcal{L}_k$, for example, $b_k = \mathcal{L}_k g + \epsilon_k$ for some $g \in h^{(i)}(\{A_n\};\Omega)$.) If $\lambda > 0$, there exists one and only one spline $S = \sum_{k=1}^M \alpha_k\, l_k \in$ $\mathrm{Spline}_{h^{(i)}(\{A_n\};\Omega)}(\mathcal{L}_1,\dots,\mathcal{L}_M)$ with the coefficient vector $\alpha = (\alpha_1,\dots,\alpha_M)^T \in \mathbb{R}^M$ which minimizes the functional*

$$\mu_\lambda(\alpha) := \sum_{k=1}^M (\mathcal{L}_k S - b_k)^2 + \lambda\, \|S\|^2_{h^{(i)}(\{A_n\};\Omega)}\ . \tag{40}$$

If $\mathcal{L}_1,\dots,\mathcal{L}_M$ are linearly independent, there exists also a uniquely determined spline S which minimizes μ_λ, if $\lambda = 0$. In both cases, the coefficient vector $\alpha = (\alpha_1,\dots,\alpha_M)^T$ is the solution of the linear system

$$\left(\sum_{k=1}^M \alpha_k\, (l_k, l_j)_{h^{(i)}(\{A_n\};\Omega)}\right) + \lambda\, \alpha_j = b_j\,, \qquad j = 1,\dots M\,. \tag{41}$$

Proof. The proof is completely analogous to the case of scalar spherical splines, which can, for example, be found in [10]. ■

Theorem 4.8 describes the situation in Problem 4.7 and shows what happens. For $\lambda = 0$ we get the interpolation problem, and if $\lambda > 0$ we perform spline smoothing. The larger λ becomes, the more weight is put on the 'smoothness' of the solution and the fewer weight is put on approximating the given data.

Furthermore, it should be noted that, in case of data $b_k = \mathcal{L}_k g = (g, l_k)_{h^{(i)}(\{A_n\};\Omega)}$, $k = 1,\dots,M$, for some fixed $g \in h^{(i)}(\{A_n\};\Omega)$, we get for $\lambda = 0$ an orthogonal projection problem: The identity (41) reads

$$\sum_{k=1}^M \alpha_k\, (l_k, l_j)_{h^{(i)}(\{A_n\};\Omega)} = (g, l_j)_{h^{(i)}(\{A_n\};\Omega)}\,, \qquad j = 1,\dots,M\,,$$

and the spline interpolation operator is just the orthogonal projector onto the space $\mathrm{Spline}_{h^{(i)}(\{A_n\};\Omega)}(\mathcal{L}_1,\dots,\mathcal{L}_M)$. The choice of the smoothing parameter λ depends, of course, on the given data. In case of noisy data or if the matrix $((l_k, l_j)_{h^{(i)}(\{A_n\};\Omega)})_{k,j=1,\dots,M}$ is extremely ill-conditioned or singular (e.g., in the case of linear dependence of $\mathcal{L}_1,\dots,\mathcal{L}_M$) we choose $\lambda > 0$. It can be verified that the functional μ_λ, given by (40) has, in fact, the structure of a Tikhonov functional (see [16]), and a smoothing parameter λ can be chosen as the parameter corresponding to the corner point of the L-curve

$$\lambda \mapsto \left(\log\left(\sum_{k=1}^M (\mathcal{L}_k S - b_k)^2\right), \log\left(\|S\|^2_{h^{(i)}(\{A_n\};\Omega)}\right)\right)\ .$$

4.3 Examples

Now we discuss the representation of the bounded linear functionals, which occur in Problems 4.3, 4.6, and 4.7, by elementary functions in certain classes of spaces. In these cases, the matrix entries of the matrix of the spline approximation problems become also available as elementary functions. This allows an easy evaluation of such a spline and an easy computation of the matrix entries. Furthermore, we will see that the representers of our bounded linear functionals are in these cases strongly space-localizing functions. This feature enables local modelling from only locally given data.

In Problem 4.3, we assume that $\{A_n\}_{n\in\mathbb{N}_0} \subset \mathbb{R}^+$ satisfies (16) and we compute a spline relative to evaluation functionals. An evaluation functional

$$\mathcal{L} : \mathcal{H}(\{A_n\};\Omega) \to \mathbb{R}\,, \quad F \mapsto \mathcal{L}F := F(\xi)\,,$$

with $\xi \in \Omega$ is fixed has, according to Lemma 3.9, the representer

$$L(\eta) = K(\xi,\eta) = \sum_{n=0}^{\infty} \frac{1}{A_n^2}\, Y_{n,j}(\xi)\, Y_{n,j}(\eta) = \sum_{n=0}^{\infty} \frac{(2n+1)}{4\pi\, A_n^2}\, P_n(\xi\cdot\eta)\,. \tag{42}$$

The matrix entries of the linear system (25) are of the type

$$K(\xi,\eta) = \sum_{n=0}^{\infty}\sum_{j=1}^{2n+1} \frac{1}{A_n^2}\, Y_{n,j}(\xi)\, Y_{n,j}(\eta) = \sum_{n=0}^{\infty} \frac{(2n+1)}{4\pi\, A_n^2}\, P_n(\xi\cdot\eta)\,, \tag{43}$$

where $\xi \in \{\eta_1^M,\ldots,\eta_M^M\}$, $\eta \in \{\xi_1^N,\ldots,\xi_N^N\}$, and K is the reproducing kernel of $\mathcal{H}(\{A_n\};\Omega)$. Thus, it suffices to find a space $\mathcal{H}(\{A_n\};\Omega)$ whose reproducing kernel (43) is available as an elementary function.

Example 4.9 *Let $\{A_n\}_{n\in\mathbb{N}_0}$ be given by $A_n := (n+\frac{1}{2})^{1/2}\, h^{-n/2}$, where $h \in (0,1)$. Then the kernel K, given by (43), is a kernel of singularity-type and has the representation (cf. [9], [10])*

$$K(\xi,\eta) = \frac{1}{2\pi}\sum_{n=0}^{\infty} h^n\, P_n(\xi\cdot\eta) = \frac{1}{2\pi}\,\frac{1}{(1+h^2-2\,h\,\xi\cdot\eta)^{1/2}}\,. \tag{44}$$

Example 4.10 *Let $\{A_n\}_{n\in\mathbb{N}_0}$ be given by $A_n := h^{-n/2}$ with $h \in (0,1)$. Then the kernel K, given by (43), is a kernel of Abel-Poisson-type and has the representation (cf. [10], [9])*

$$K(\xi,\eta) = \sum_{n=0}^{\infty} \frac{(2n+1)}{4\pi}\, h^n\, P_n(\xi\cdot\eta) = \frac{1}{4\pi}\,\frac{1-h^2}{(1+h^2-2\,h\,\xi\cdot\eta)^{3/2}}\,. \tag{45}$$

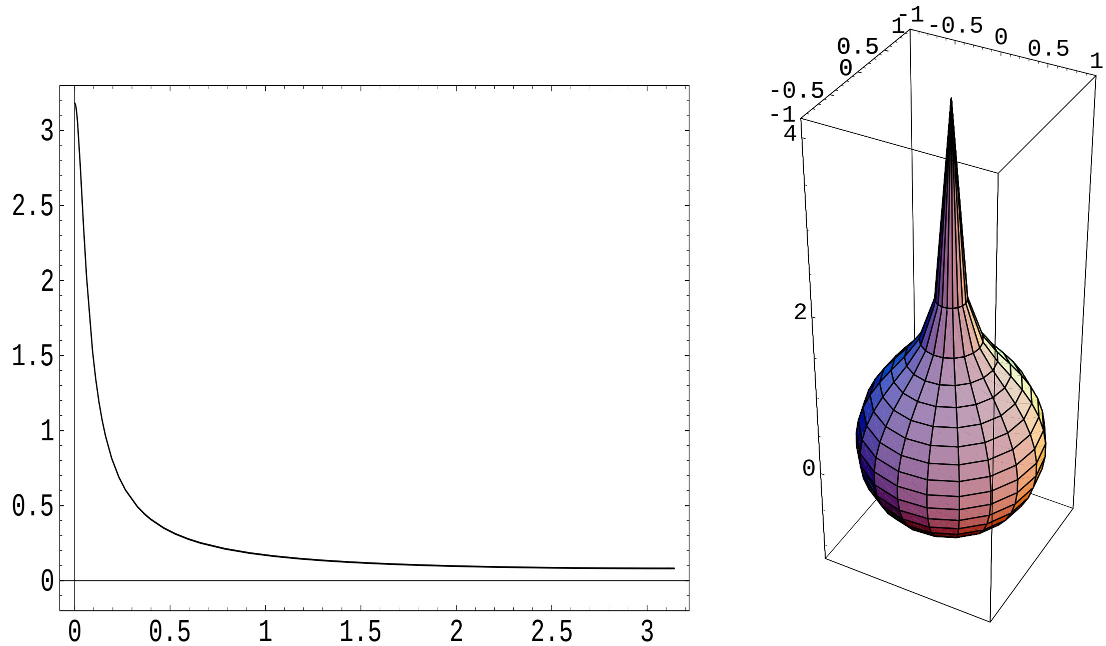


Figure 1: *The scalar spline L (representer of the evaluation functional) in Example 4.9 for $\xi = (0,0,1)^T$ and $h = 0.95$. The spline L is rotationally symmetric, i.e., $L(\varphi_1, \vartheta) = L(\varphi_2, \vartheta)$ for all $\varphi_1, \varphi_2 \in [0, 2\pi)$ and all $\vartheta \in [0, \pi]$, and the left picture shows L for a fixed φ as function of ϑ. The right picture shows the spline L modelled on the unit sphere.*

The choices of $\{A_n\}_{n\in\mathbb{N}_0}$ as in Example 4.9 and 4.10 also allow an elementary representation of L^*S, because

$$L^*S = L^*\left(\sum_{k=1}^{N} \alpha_k\, L_k\right) = \sum_{k=1}^{N} \alpha_k\, L^*L_k$$

and

$$L^*_\eta L_k(\eta) = L^*_\eta K(\eta_k^M, \eta)\,, \qquad \eta \in \Omega\,.$$

As K depends only on the inner product of its two arguments, i.e., K can be regarded as a function in $\widetilde{K} \in \mathcal{C}([-1,1])$, via $\widetilde{K}(\xi\cdot\eta) := K(\xi,\eta)$ for $\xi,\eta \in \Omega$. According to (6),

$$L^*_\eta K(\xi,\eta) = L^*_\eta \widetilde{K}(\xi\cdot\eta) = \widetilde{K}'(\xi\cdot\eta)\,(\eta\wedge\xi)\,, \qquad \xi,\eta \in \Omega\,.$$

Clearly, a representation of K by an elementary function implies that $L^*_\eta K(\xi,\eta)$ is available as an elementary function.

In Problem 4.6 we assume that $\{A_n\}_{n\in\mathbb{N}_0} \subset \mathbb{R}^+$ satisfies (16). We compute a spline relative to bounded linear functionals of the type

$$\mathcal{L} : h^{(3)}(\{A_n\};\Omega) \to \mathbb{R}\,, \quad f \mapsto \mathcal{L}f := O^{(3)}_\xi f(\xi), \tag{46}$$

where $\xi \in \Omega$ is fixed. According to Lemma 3.11, $\mathcal{L}$ has the representer

$$l(\eta) = \sum_{n=1}^{\infty}\sum_{j=1}^{2n+1} \frac{1}{A_n^2}\,(n(n+1))^{1/2}\, Y_{n,j}(\xi)\, y^{(3)}_{n,j}(\eta)\,, \qquad \eta \in \Omega\,. \tag{47}$$

In order to get a different representation of l, note that

$$K(\xi,\cdot) = \sum_{n=0}^{\infty}\sum_{j=1}^{2n+1} \frac{1}{A_n^2}\, Y_{n,j}(\xi)\, Y_{n,j} \;\in \mathcal{H}(\{A_n\};\Omega)$$

with respect to the second variable. In particular, $K(\xi,\cdot)$ may be differentiated term by term:

$$\begin{aligned} o_\eta^{(3)} K(\xi,\eta) &= \sum_{n=1}^{\infty}\sum_{j=1}^{2n+1} \frac{1}{A_n^2}\, Y_{n,j}(\xi)\, o_\eta^{(3)} Y_{n,j}(\eta) \\ &= \sum_{n=1}^{\infty}\sum_{j=1}^{2n+1} \frac{1}{A_n^2}\, (n(n+1))^{1/2}\, Y_{n,j}(\xi)\, y_{n,j}^{(3)}(\eta)\,, \qquad \eta\in\Omega\,. \end{aligned}$$

Hence, l given by (47) can be written $l(\eta) = L_\eta^* K(\xi,\eta)$, $\eta\in\Omega$, with

$$K(\xi,\eta) = \frac{1}{4\pi}\sum_{n=0}^{\infty} \frac{(2n+1)}{A_n^2}\, P_n(\xi\cdot\eta)\,, \qquad \xi,\eta\in\Omega\,. \tag{48}$$

The kernel K depends, in fact, only on the inner product $\xi\cdot\eta$ and can, therefore, be regarded as a function $\widetilde{K}\in\mathcal{C}^{(1)}([-1,1])$, where $\widetilde{K}(\xi\cdot\eta) := K(\xi,\eta)$ for $\xi,\eta\in\Omega$. According to (6),

$$L_\eta^* K(\xi,\eta) = L_\eta^* \widetilde{K}(\xi\cdot\eta) = \widetilde{K}'(\xi\cdot\eta)\,(\eta\wedge\xi)\,, \qquad \xi,\eta\in\Omega\,.$$

Consequently, with K given as in (48)

$$l(\eta) = L_\eta^* K(\xi,\eta) = L_\eta^* \widetilde{K}(\xi\cdot\eta) = \widetilde{K}'(\xi\cdot\eta)\,(\eta\wedge\xi)\,, \qquad \eta\in\Omega\,. \tag{49}$$

Hence, we get a representation of l by an elementary function, if $K(\xi,\eta) = \widetilde{K}(\xi\cdot\eta)$ is available as an elementary function.

A spline in Problem 4.6 is a linear combination of functions of type (49), and the entries of the matrix in Problem 4.6 (a) or (b) are of the type $a\cdot l(\eta)$, where $a\in\mathbb{R}^3$, $|a|=1$. Thus, availability of l as an elementary function leads to availability of the matrix entries as elementary functions.

Next, we give two examples of a choice of $\{A_n\}_{n\in\mathbb{N}_0}$ satisfying (16), which leads to an elementary representation of the kernel K in (48). We compute the representer l of $\mathcal{L}$, given by (46), for these cases.

Example 4.11 *Let $\{A_n\}_{n\in\mathbb{N}_0}$ be given by $A_n := (n+\frac{1}{2})^{1/2}\, h^{-n/2}$ with $h\in(0,1)$. Then the kernel K in (48) is a kernel of singularity-type and has the representation (44). The representer l, given by (49), reads*

$$l(\eta) = \frac{1}{2\pi}\,\frac{h}{(1+h^2-2\,h\,\xi\cdot\eta)^{3/2}}\,(\eta\wedge\xi)\,, \qquad \eta\in\Omega\,.$$

Example 4.12 *Let $\{A_n\}_{n\in\mathbb{N}_0}$ be given by $A_n := h^{-n/2}$ with $h\in(0,1)$. Then the kernel K in (48) is a kernel of Abel-Poisson-type and has the representation (45). The representer l, given by (49), reads*

$$l(\eta) = \frac{1}{4\pi}\,\frac{3\,h\,(1-h^2)}{(1+h^2-2\,h\,\xi\cdot\eta)^{5/2}}\,(\eta\wedge\xi)\,, \qquad \eta\in\Omega\,.$$

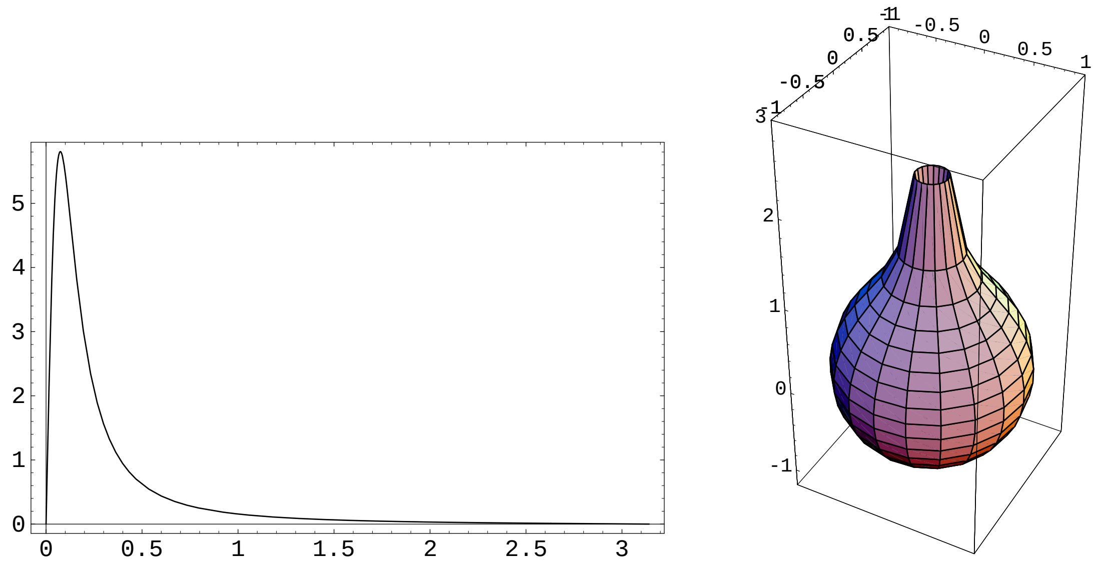


Figure 2: *Norm of the vector spline* l *in Example 4.11 for* $\xi = (0,0,1)^T$, $h = 0.9$. *The norm of* l *is rotationally symmetric, i.e.,* $l(\varphi_1, \vartheta) = l(\varphi_2, \vartheta)$ *for all* $\varphi_1, \varphi_2 \in [0, 2\pi)$ *and all* $\vartheta \in [0, \pi]$, *and the left picture shows the norm of* l *for a fixed* φ *as function of* ϑ. *The right picture shows the norm of* l *on the unit sphere, where all function values are multiplied by* $1/3$ *for reasons of a better illustration.*

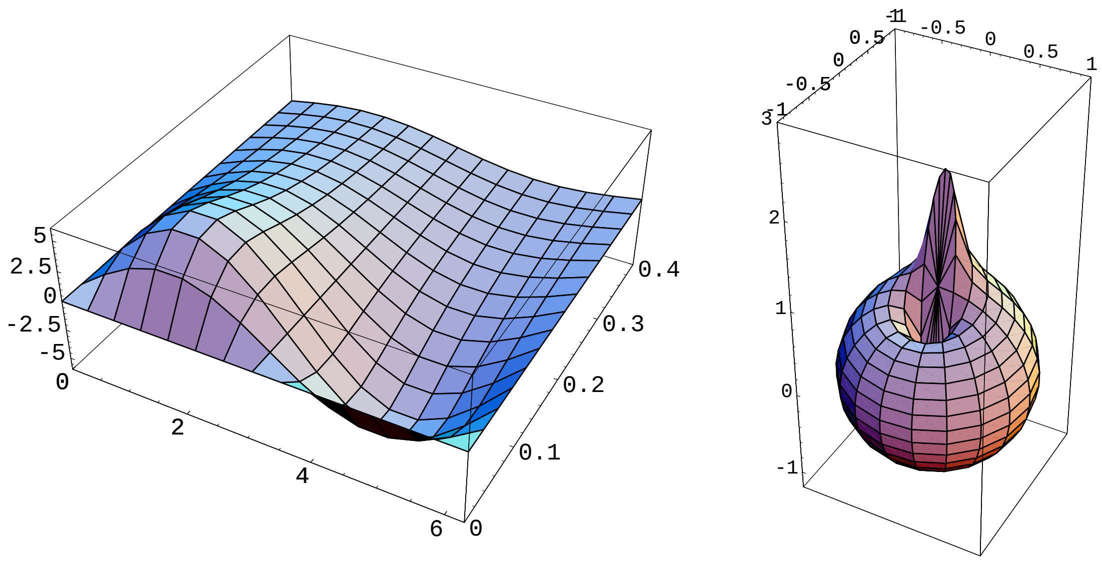


Figure 3: *First Euclidean component of the vector spline* l *in Example 4.11 for* $\xi = (0,0,1)^T$, $h = 0.9$. *The left picture shows the first Euclidean component of* l *in the parameterization* $\eta = (\cos(\varphi)\sin(\vartheta), \sin(\varphi)\sin(\vartheta), \cos(\vartheta))^T$ *in the* (φ, ϑ)*-plane for* $0 \le \varphi \le 2\pi$, $0 \le \vartheta \le 0.4$. *The right picture shows the first Euclidean component of* l *on the unit sphere, where all function values are multiplied by* $1/3$ *for reasons of a better illustration.*

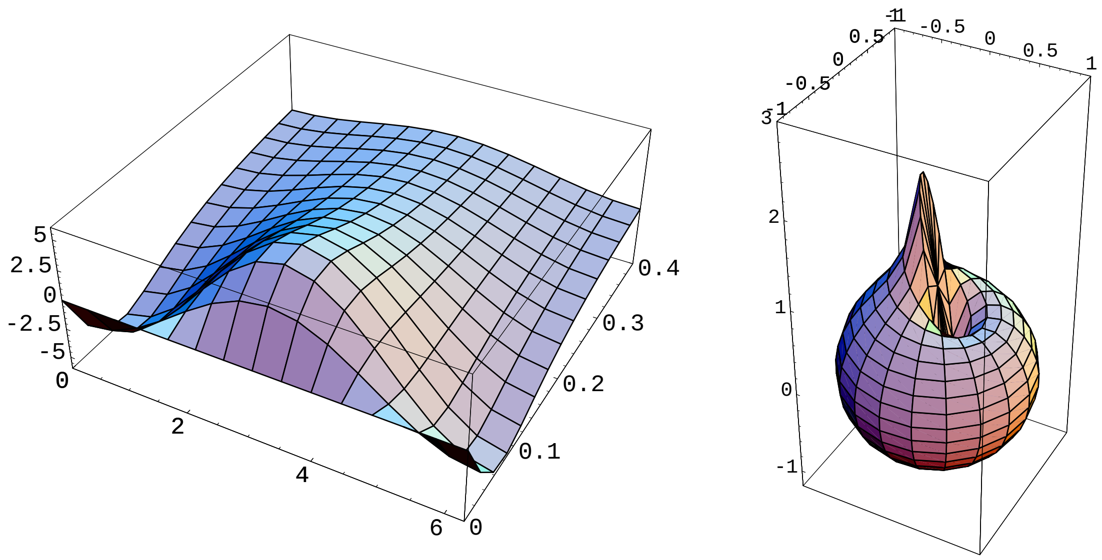


Figure 4: *Second Euclidean component of the vector spline l in Example 4.11 for $\xi = (0,0,1)^T$, $h = 0.9$. The left picture shows the second Euclidean component of l in the parameterization $\eta = (\cos(\varphi)\,\sin(\vartheta), \sin(\varphi)\,\sin(\vartheta), \cos(\vartheta))^T$ in the (φ,ϑ)-plane for $0 \le \varphi \le 2\pi$, $0 \le \vartheta \le 0.4$. The right picture shows the second Euclidean component of l on the unit sphere, where all function values are multiplied by $1/3$ for reasons of a better illustration.*

The third Euclidean component of the vector spline l in Example 4.11 with $\xi = (0,0,1)^T$ vanishes because the third Euclidean component of the vector product $\eta \wedge (0,0,1)^T$ is zero for all $\eta \in \Omega$.

Now, we come to Problem 4.7. In Problem 4.7 it is assumed that $\{A_n\}_{n\in\mathbb{N}_0} \subset \mathbb{R}^+$ satisfies (10). We have to consider a spline relative to bounded linear functionals of the type

$$\mathcal{L} : h^{(3)}(\{A_n\};\Omega) \to \mathbb{R}, \quad f \mapsto \mathcal{L}f := a \cdot f(\xi)\,, \tag{50}$$

where $\xi \in \Omega$ and $a \in \mathbb{R}^3$, $|a| = 1$, are fixed. Due to Lemma 3.12, $\mathcal{L}$ has the representer l given by

$$l(\eta) = \sum_{n=1}^{\infty} \sum_{j=1}^{2n+1} \frac{1}{A_n^2} \left(a \cdot y_{n,j}^{(3)}(\xi) \right) y_{n,j}^{(3)}(\eta)\,, \qquad \eta \in \Omega\,. \tag{51}$$

Because of condition (10) and Theorem 3.2, the kernel

$$K(\xi,\eta) = \sum_{n=0}^{\infty} \sum_{j=1}^{2n+1} \frac{1}{A_n^2\, n(n+1)}\, Y_{n,j}(\xi)\, Y_{n,j}(\eta)\,, \qquad \xi,\eta \in \Omega\,, \tag{52}$$

satisfies $K(\xi,\cdot) \in \mathcal{H}(\{A_n(n+\frac{1}{2})\};\Omega) \subset \mathcal{C}^{(1)}(\Omega)$. In particular, $K(\xi,\cdot)$ may be differentiated term by term

$$o_\eta^{(3)} K(\xi,\eta) = \sum_{n=0}^{\infty} \sum_{j=1}^{2n+1} \frac{1}{A_n^2\, (n(n+1))^{1/2}}\, Y_{n,j}(\xi)\, y_{n,j}^{(3)}(\eta)\,, \qquad \eta \in \Omega\,.$$

$o^{(3)}_\eta K(\cdot,\eta)$ is a function in $h^{(3)}(\{A_n(n+\frac{1}{2})\};\Omega)$, which may be differentiated (componentwise) term by term. Consequently, for $i=1,2,3$

$$o^{(3)}_\xi\left(\left(o^{(3)}_\eta K(\xi,\eta)\right)\cdot\varepsilon^i\right)=\sum_{n=1}^{\infty}\sum_{j=1}^{2n+1}\frac{1}{A_n^2}\,y^{(3)}_{n,j}(\xi)\,\left(\varepsilon^i\cdot y^{(3)}_{n,j}(\eta)\right),$$

and

$$\sum_{i=1}^{3}\left(a\cdot\left(o^{(3)}_\xi\left(\left(o^{(3)}_\eta K(\xi,\eta)\right)\cdot\varepsilon^i\right)\right)\right)\varepsilon^i=\sum_{n=1}^{\infty}\sum_{j=1}^{2n+1}\frac{1}{A_n^2}\left(a\cdot y^{(3)}_{n,j}(\xi)\right)y^{(3)}_{n,j}(\eta)\,.$$

Thus l, given by (51), has the representation

$$l(\eta)=\sum_{i=1}^{3}\left(a\cdot\left(o^{(3)}_\xi\left(\left(o^{(3)}_\eta K(\xi,\eta)\right)\cdot\varepsilon^i\right)\right)\right)\varepsilon^i\,,\qquad \eta\in\Omega\,, \tag{53}$$

and (53) is available as an elementary function, whenever K, given by (52), is an elementary function. The matrix entries in Problem 4.7 are inner products $(l_1,l_2)_{h^{(3)}(\{A_n\};\Omega)}$ of two representers l_1,l_2 of bounded linear functionals $\mathcal{L}_1$ and $\mathcal{L}_2$, respectively, which are of type (50), i.e., $\mathcal{L}_i f:=a_i\cdot f(\xi_i)$. It can be easily verified that

$$(l_1,l_2)_{h^{(3)}(\{A_n\};\Omega)}=\sum_{i=1}^{3}(a_2\cdot\varepsilon^i)\left(a_1\cdot\left(o^{(3)}_{\xi_1}\left(\left(o^{(3)}_{\xi_2}K(\xi_1,\xi_2)\right)\cdot\varepsilon^i\right)\right)\right)=a_2\cdot l_1(\xi_2)\,. \tag{54}$$

The kernel K, given by (52), depends only on the inner product $\xi\cdot\eta$, i.e.,

$$K(\xi,\eta)=\frac{1}{4\pi}\sum_{n=0}^{\infty}\frac{(2n+1)}{A_n^2\,n(n+1)}\,P_n(\xi\cdot\eta)\,,\qquad \xi,\eta\in\Omega\,,$$

and can be interpreted as a function $\widetilde{K}\in\mathcal{C}^{(2)}([-1,1])$, via $\widetilde{K}(\xi\cdot\eta):=K(\xi,\eta)$ for $\xi,\eta\in\Omega$. Thus, we get

$$o^{(3)}_\eta K(\xi,\eta)=L^*_\eta\widetilde{K}(\xi\cdot\eta)=\widetilde{K}'(\xi\cdot\eta)\,(\eta\wedge\xi)\,,\qquad \xi,\eta\in\Omega\,,$$

and l, given by (51), reads

$$\begin{aligned} l(\eta) &= \sum_{i=1}^{3}\left(a\cdot\left(L^*_\xi\left(\left(\widetilde{K}'(\xi\cdot\eta)\,(\eta\wedge\xi)\right)\cdot\varepsilon^i\right)\right)\right)\varepsilon^i \\ &= \sum_{i=1}^{3}\left(a\cdot\left(\widetilde{K}''(\xi\cdot\eta)\,\left((\eta\wedge\xi)\cdot\varepsilon^i\right)\,(\xi\wedge\eta)+\widetilde{K}'(\xi\cdot\eta)\,L^*_\xi\left((\eta\wedge\xi)\cdot\varepsilon^i\right)\right)\right)\varepsilon^i\,. \end{aligned}$$

In order to simplify this term, we use that

$$(\eta\wedge\xi)\cdot\varepsilon^i=(\varepsilon^i\wedge\eta)\cdot\xi$$

and

$$L^*_\xi\left((\eta\wedge\xi)\cdot\varepsilon^i\right)=L^*_\xi\left(\xi\cdot(\varepsilon^i\wedge\eta)\right)=\xi\wedge(\varepsilon^i\wedge\eta)\,.$$

Hence,

$$\begin{aligned} l(\eta) &= \sum_{i=3}^{3}\left(a\cdot(\xi\wedge\eta)\right)\,\widetilde{K}''(\xi\cdot\eta)\,\left((\eta\wedge\xi)\cdot\varepsilon^i\right)\,\varepsilon^i \\ &+ \sum_{i=1}^{3}\widetilde{K}'(\xi\cdot\eta)\left(a\cdot\left(\xi\wedge(\varepsilon^i\wedge\eta)\right)\right)\,\varepsilon^i\,. \end{aligned}$$

Because of

$$a \cdot (\xi \wedge (\varepsilon^i \wedge \eta)) = (\varepsilon^i \wedge \eta) \cdot (a \wedge \xi) = \varepsilon^i \cdot (\eta \wedge (a \wedge \xi))$$

we get

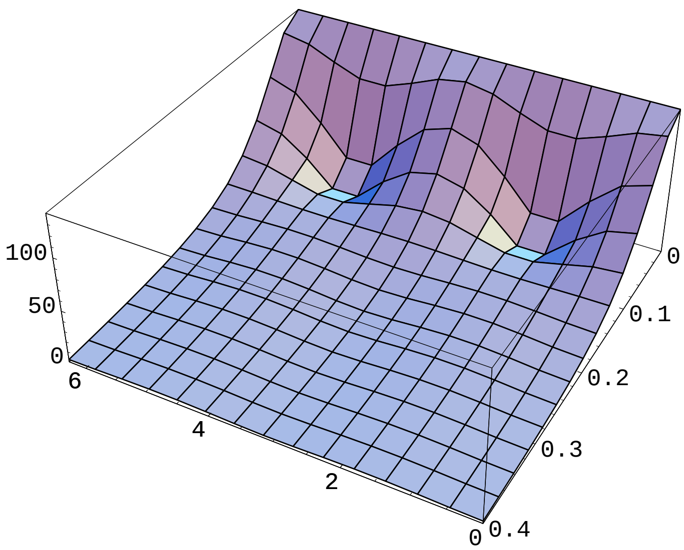
$$l(\eta) = (a \cdot (\xi \wedge \eta))\, \widetilde{K}''(\xi \cdot \eta)\, (\eta \wedge \xi) + \widetilde{K}'(\xi \cdot \eta)\, (\eta \wedge (a \wedge \xi)) .$$

A matrix entry (54) reads then

$$\begin{aligned} a_2 \cdot l_1(\xi_2) \quad &= \quad (a_2 \cdot (\xi_2 \wedge \xi_1))\, (a_1 \cdot (\xi_1 \wedge \xi_2))\, \widetilde{K}''(\xi_1 \cdot \xi_2) \\ &\quad + \widetilde{K}'(\xi_1 \cdot \xi_2)\, (a_2 \cdot (\xi_2 \wedge (a_1 \wedge \xi_1))) \,. \end{aligned}$$

Example 4.13 *Let $\{A_n\}_{n \in \mathbb{N}_0} \subset \mathbb{R}^+$ be given by $A_n := h^{-n/2}\, (n + \frac{1}{2})^{1/2}\, (n(n+1))^{-1/2}$, where $h \in (0,1)$. Then the kernel K, given in (52), is a kernel of singularity-type with the representation (44), and the representer l of the bounded linear functional $\mathcal{L}$, given by (50), reads*

$$\begin{aligned} l(\eta) \quad &= \quad (a \cdot (\xi \wedge \eta))\, \frac{1}{2\pi}\, \frac{3\,h^2}{(1 + h^2 - 2\,h\,\xi \cdot \eta)^{5/2}}\, (\eta \wedge \xi) \\ &\quad + \frac{1}{2\pi}\, \frac{h}{(1 + h^2 - 2\,h\,\xi \cdot \eta)^{3/2}}\, (\eta \wedge (a \wedge \xi))\,, \qquad \eta \in \Omega\,. \end{aligned}$$



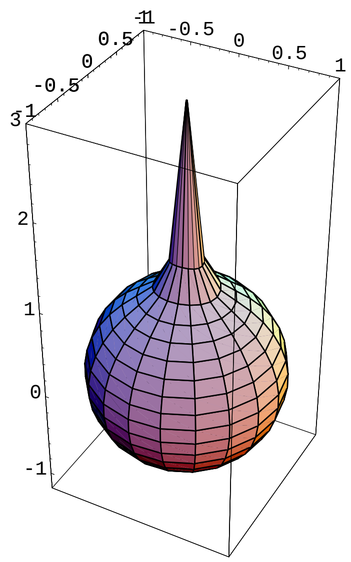


Figure 5: *Norm of the vector spline l in Example 4.13 for $\xi = (0,0,1)^T$, $a = (1,0,0)^T$, and $h = 0.9$. The left picture shows the norm of the vector spline l in the parameterization $\eta = (\cos(\varphi)\, \sin(\vartheta), \sin(\varphi)\, \sin(\vartheta), \cos(\vartheta))^T$ in the (φ, ϑ)-plane for $0 \le \varphi \le 2\pi$, $0 \le \vartheta \le 0.4$. The right picture shows the norm of l on the unit sphere, where all function values are multiplied by 1/70 for means of a better illustration.*

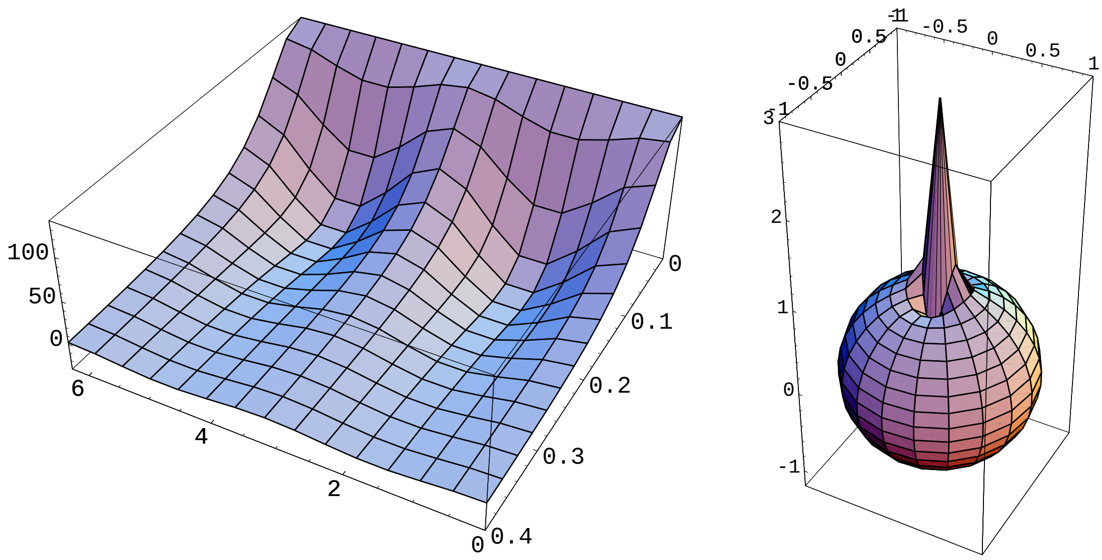


Figure 6: *First Euclidean component of the vector spline l in Example 4.13 for $\xi = (0,0,1)^T$, $a = (1,0,0)^T$, and $h = 0.9$. The left picture shows the first Euclidean component of l in the parameterization $\eta = (\cos(\varphi)\,\sin(\vartheta), \sin(\varphi)\,\sin(\vartheta), \cos(\vartheta))^T$ in the (φ,ϑ)-plane for $0 \le \varphi \le 2\pi$, $0 \le \vartheta \le 0.4$. The right picture shows the first Euclidean component of l on the unit sphere, where all function values are multiplied by $1/70$ for means of a better illustration.*

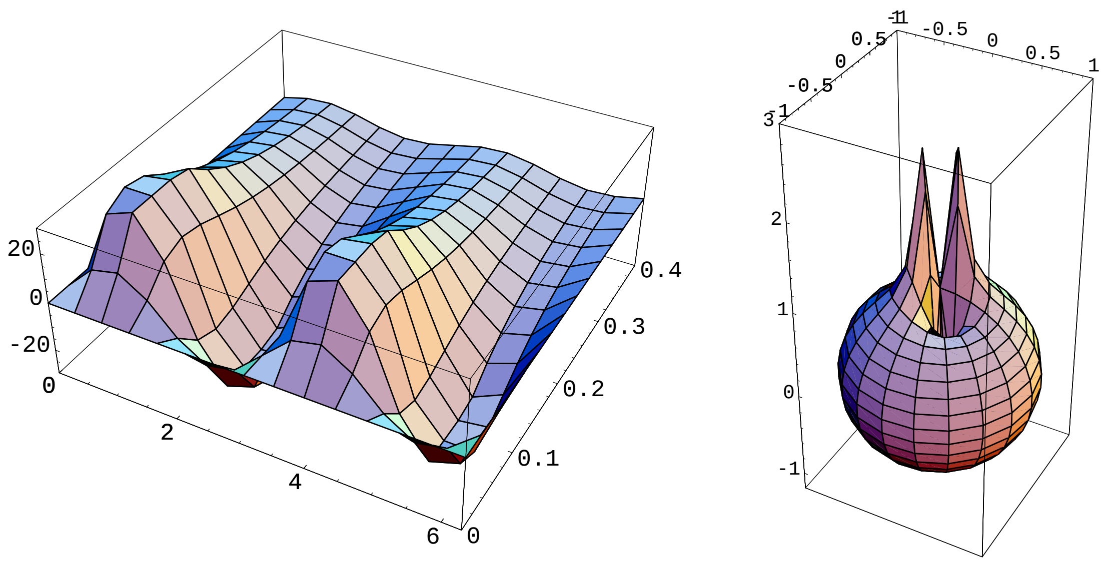


Figure 7: *Second Euclidean component of the vector spline l in Example 4.13 for $\xi = (0,0,1)^T$, $a = (1,0,0)^T$, and $h = 0.9$. The left picture shows the second Euclidean component of l in the parameterization $\eta = (\cos(\varphi)\,\sin(\vartheta), \sin(\varphi)\,\sin(\vartheta), \cos(\vartheta))^T$ in the (φ,ϑ)-plane for $0 \le \varphi \le 2\pi$, $0 \le \vartheta \le 0.4$. The right picture shows the second Euclidean component of l on the unit sphere, where all function values are multiplied by $1/15$ for means of a better illustration.*

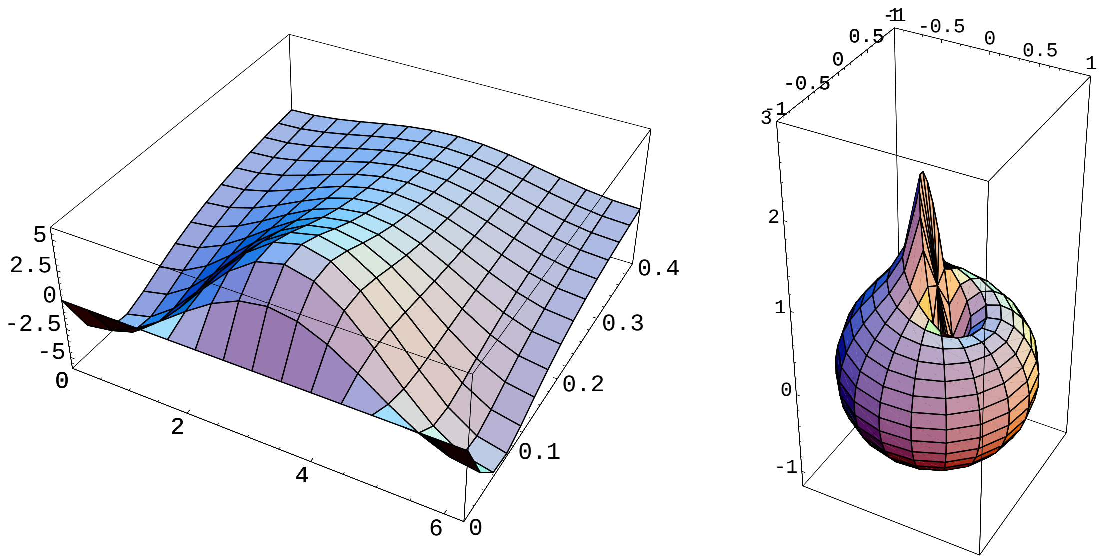


Figure 8: *Third Euclidean component of the vector spline l in Example 4.13 for $\xi = (0,0,1)^T$, $a = (1,0,0)^T$, and $h = 0.9$. The left picture shows the third Euclidean component of l in the parameterization $\eta = (\cos(\varphi)\,\sin(\vartheta), \sin(\varphi)\,\sin(\vartheta), \cos(\vartheta))^T$ in the (φ,ϑ)-plane for $0 \le \varphi \le 2\pi$, $0 \le \vartheta \le 0.4$. The right picture shows the third Euclidean component of l on the unit sphere, where all function values are multiplied by $1/3$ for means of a better illustration.*

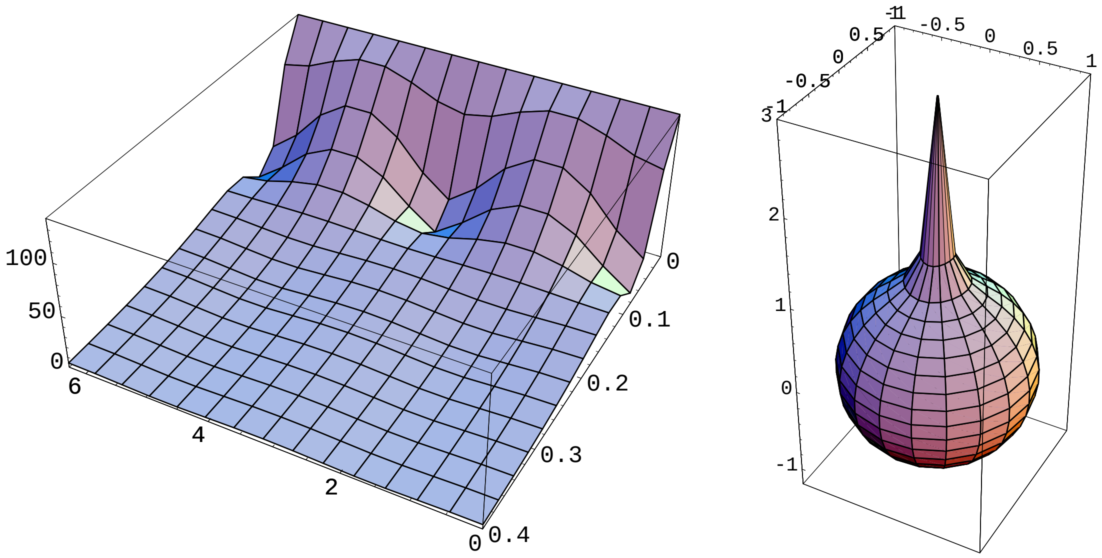


Figure 9: *Norm of the vector spline l in Example 4.13 for $\xi = (0,0,1)^T$, $a = (0,1,0)^T$, and $h = 0.9$. The left picture shows the norm of the vector spline l in the parameterization $\eta = (\cos(\varphi)\,\sin(\vartheta), \sin(\varphi)\,\sin(\vartheta), \cos(\vartheta))^T$ in the (φ,ϑ)-plane for $0 \le \varphi \le 2\pi$, $0 \le \vartheta \le 0.4$. The right picture shows the norm of l on the unit sphere, where all function values are multiplied by $1/70$ for means of a better illustration.*

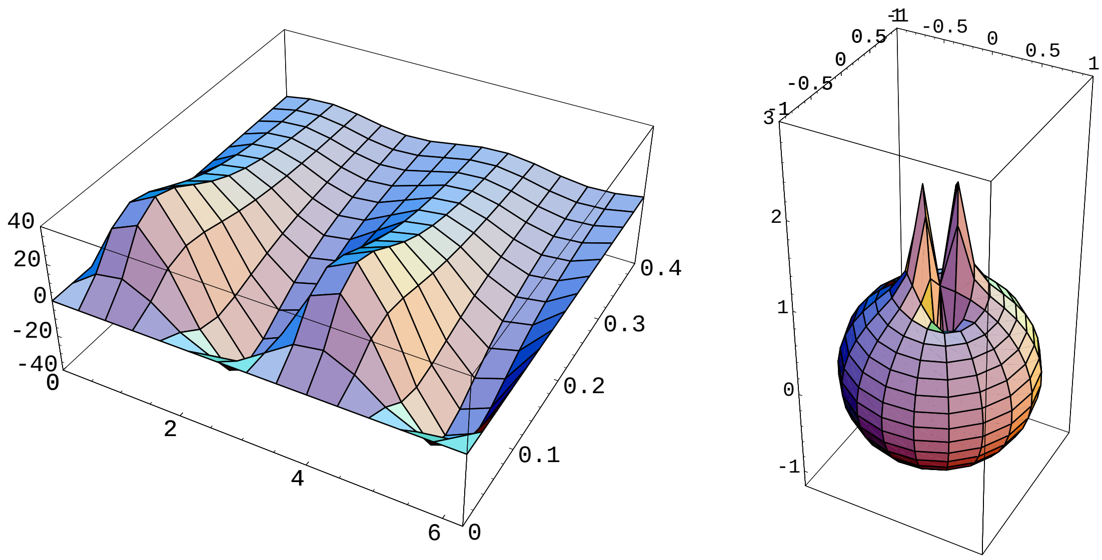


Figure 10: *First Euclidean component of the vector spline l in Example 4.13 for* $\xi = (0,0,1)^T$, $a = (0,1,0)^T$, *and* $h = 0.9$. *The left picture shows the first Euclidean component of l in the parameterization* $\eta = (\cos(\varphi)\,\sin(\vartheta), \sin(\varphi)\,\sin(\vartheta), \cos(\vartheta))^T$ *in the (φ, ϑ)-plane for* $0 \le \varphi \le 2\pi$, $0 \le \vartheta \le 0.4$. *The right picture shows the first Euclidean component of l on the unit sphere, where all function values are multiplied by* $1/20$ *for means of a better illustration.*

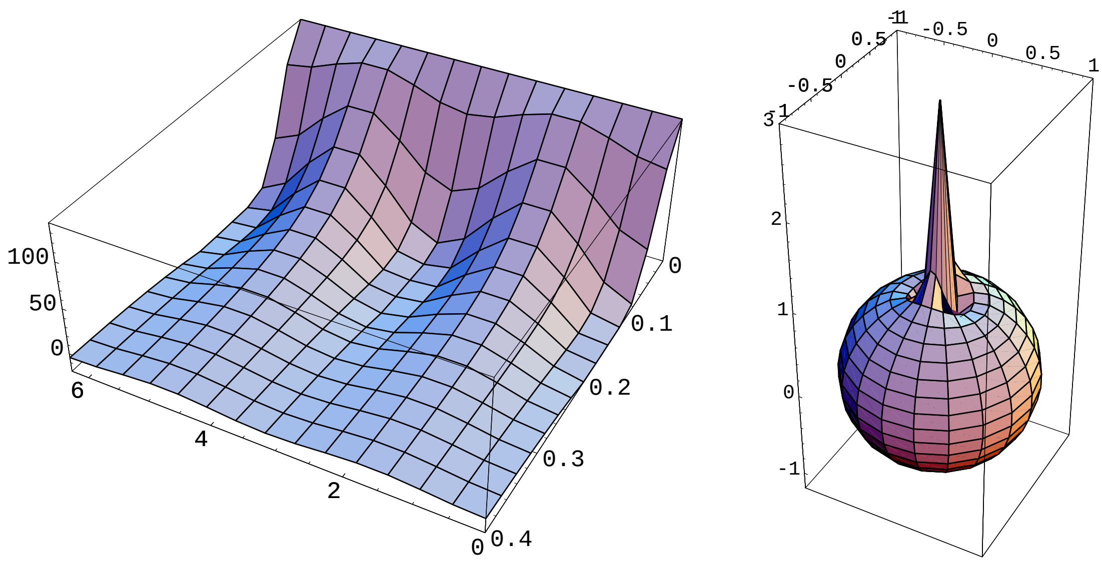


Figure 11: *Second Euclidean component of the vector spline l in Example 4.13 for* $\xi = (0,0,1)^T$, $a = (0,1,0)^T$, *and* $h = 0.9$. *The left picture shows the second Euclidean component of l in the parameterization* $\eta = (\cos(\varphi)\,\sin(\vartheta), \sin(\varphi)\,\sin(\vartheta), \cos(\vartheta))^T$ *in the (φ, ϑ)-plane for* $0 \le \varphi \le 2\pi$, $0 \le \vartheta \le 0.4$. *The right picture shows the second Euclidean component of l on the unit sphere, where all function values are multiplied by* $1/70$ *for means of a better illustration.*

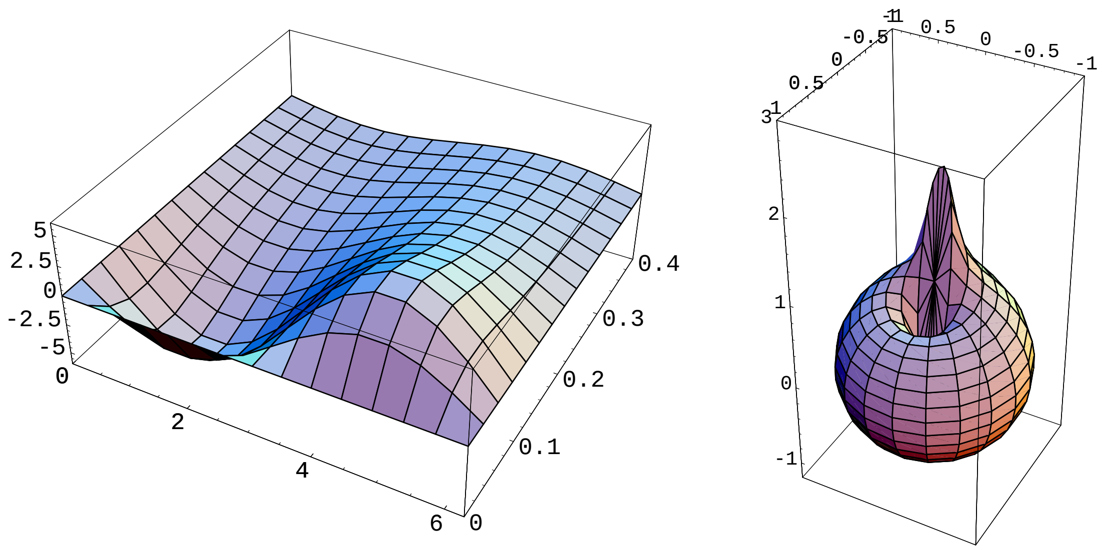


Figure 12: *Third Euclidean component of the vector spline l in Example 4.13 for $\xi = (0,0,1)^T$, $a = (0,1,0)^T$, and $h = 0.9$. The left picture shows the third Euclidean component of l in the parameterization $\eta = (\cos(\varphi)\,\sin(\vartheta), \sin(\varphi)\,\sin(\vartheta), \cos(\vartheta))^T$ in the (φ,ϑ)-plane for $0 \le \varphi \le 2\pi$, $0 \le \vartheta \le 0.4$. The right picture shows the third Euclidean component of l on the unit sphere, where all function values are multiplied by $1/3$ for means of a better illustration.*

Considering Figures 5 to 12, it should be noted that $(1,0,0)^T$ and $(0,1,0)^T$ are an orthonormal basis of the tangential space to the sphere in the point $\xi = (0,0,1)^T$. For $\xi = (0,0,1)^T$ and $a = (0,0,1)^T$, we get in Example 4.13 a vector spline l which vanishes everywhere on the sphere Ω, i.e., $l = 0$.

Example 4.14 *Let $\{A_n\}_{n\in\mathbb{N}_0} \subset \mathbb{R}^+$ be given by $A_n := h^{-n/2}(n(n+1))^{-1/2}$, where $h \in (0,1)$. Then the kernel K, given in (52), is a kernel of Abel-Poisson-type and has the representation (45). The representer l of the bounded linear functional $\mathcal{L}$, given by (50), reads*

$$\begin{aligned} l(\eta) \quad &= \quad (a\cdot(\xi\wedge\eta))\,\frac{1}{4\pi}\,\frac{15\,h^2(1-h^2)}{(1+h^2-2\,h\,\xi\cdot\eta)^{7/2}}\,(\eta\wedge\xi) \\ &\quad + \frac{1}{4\pi}\,\frac{3\,h\,(1-h^2)}{(1+h^2-2\,h\,\xi\cdot\eta)^{5/2}}\,(\eta\wedge(a\wedge\xi))\,, \qquad \xi\in\Omega\,. \end{aligned}$$

The illustrations of the representers occuring in Examples 4.9, 4.11, and 4.13 show that in all the cases the representers l are strongly space-localizing functions, which can numerically be regarded as vanishing outside a certain neighbourhood of ξ (ξ is the fixed evaluation point). This enables local modelling from only locally given data in wind field modelling. Furthermore, the parameter $h \in (0,1)$ in Examples 4.9, 4.10, 4.11, 4.12, 4.13, and 4.14 controls the space-localization of the representer l (the closer h is to 1, the stronger is the space localization). This allows us to tune the space-localization in correspondence with the density of the given data.

5 The Schwarz Alternating Algorithm: A Domain Decomposition Method

In what follows we are interested in discussing how the interpolation or smoothing problems under consideration can be solved numerically. All the spline approximation problems in the last section lead to a linear system with a positive definite symmetric matrix:

$$\mathbf{A}x = b\,,\ \text{where } \mathbf{A} \in \mathbb{R}^{N\times N},\ \mathbf{A}^T = \mathbf{A},\ (\mathbf{A}y, y) > 0 \text{ for all } y \in \mathbb{R}^N\backslash\{0\},\ \text{and } x, b \in \mathbb{R}^N. \quad (55)$$

Such a linear equation system can be solved with iterative solvers or direct solvers, but if N is large (for example, $N \geq 10000$), the runtime of an iterative solver without a suitable preconditioner or of a direct solver increases tremendously. Therefore, we need a more sophisticated method to solve (55) for problems with a large number of given data. One such method is a multiplicative variant of the Schwarz alternating algorithm, a domain decomposition method, which allows it to split the matrix $\mathbf{A}$ in (55) into several smaller submatrices, which may (and will in numerical implementations) in general overlap. This multiplicative variant of the Schwarz alternating algorithm is an iterative method which solves in each iteration step linear systems with the matrices obtained from the splitting successively. This reduces both runtime and memory requirement drastically. A further speed-up can be achieved if an additive variant of the Schwarz alternating algorithm is used which runs on parallel computers.

The Schwarz alternating algorithm dates back to H.A. Schwarz' work [25], published in 1890, and has been investigated by many authors since then. A revived interest in variants of the Schwarz alternating method arose since 1985, due to the availability of fast modern and parallel computers. Roughly speaking, there are mainly two types of the Schwarz alternating algorithm: multiplicative variants (like the one used in this paper) and additive variants, which can be implemented on parallel computers and which are usually faster. For more information about the Schwarz alternating algorithm, the reader is refereed to, for example, [19], [20], [21], [27], and [5]. In the last few years, a great interest has also been taken in the relation between the Schwarz alternating algorithm, multisplittings, multigrid methods, preconditioned conjugate gradient methods, as well as other iterative schemes (see, for example, [27], [11], [12]). The use of a multiplicative variant of the Schwarz alternating algorithm for the solution of spline interpolation or smoothing problems in Sobolev spaces $\mathcal{H}(\{A_n\};\Omega)$, $h^{(i)}(\{A_n\};\Omega)$, or $\mathcal{H}(\{A_n\};\overline{\Omega^{\text{ext}}})$ (see [1], [13], [16] for theoretical and numerical investigations of the multiplicative variant of the Schwarz alternating algorithm for spline approximation in the spaces $\mathcal{H}(\{A_n\};\overline{\Omega^{\text{ext}}})$), was initially inspired by the use of the Schwarz alternating algorithm in radial basis function interpolation (see [3]). It should, however, be noted that the paper [3] discusses only the case of radial basis function interpolation (and not of smoothing) and that splines in $h^{(i)}(\{A_n\};\Omega)$ are even componentwise no (scalar) radial basis functions.

Most of the results in this section are given with a proof, because the authors could not find a proof of the multiplicative variant of the Schwarz alternating algorithm for the general case of a linear equation system with a positive definite symmetric matrix, in the way it will be needed here.

5.1 The Multiplicative Schwarz Alternating Algorithm for Positive Definite Symmetric Matrices

The solution of (55) with a multiplicative variant of the Schwarz alternating algorithm, which will from now on be called briefly the multiplicative Schwarz alternating algorithm, is based on two facts: (i) every positive definite symmetric matrix is a Gram matrix, (ii) the convergence proof of the multiplicative Schwarz alternating algorithm is based on its formulation in terms of orthogonal projectors.

The matrix $\mathbf{A} = (A_{i,j})_{i,j=1,\ldots N}$ in (55) is positive definite and symmetric. Due to the theorem about the Cholesky factorization (see, for example, [14]), there exists an uniquely determined invertible lower triangular matrix $\mathbf{L}$ with positive diagonal entries, such that

$$\mathbf{A} = \mathbf{L}\,\mathbf{L}^T\,. \tag{56}$$

Denote the row vectors of $\mathbf{L}$ by $v_1, \ldots, v_N$. Then (56) implies that

$$A_{ij} = v_i \cdot v_j = (v_i, v_j)\,, \qquad i,j = 1, \ldots, N\,.$$

Thus, $\mathbf{A}$ is the Gram matrix of the basis $\{v_1, \ldots, v_N\}$ of $\mathbb{R}^N$, and the solution $x = (x_1, \ldots, x_N)^T$ of the linear system (55) is the solution of the following orthogonal projection problem: Find $x = (x_1, \ldots, x_N)^T \in \mathbb{R}^N$ such that $f \in \mathbb{R}^N$ with $(f, v_i) = b_i$, $i = 1, \ldots, N$, has the representation

$$f = \sum_{i=1}^{N} x_i\, v_i\,. \tag{57}$$

Indeed, the solution of this problem demands the solution of the linear system

$$\sum_{i=1}^{N} x_i\,(v_i, v_j) = (f, v_j) = b_j\,, \qquad j = 1, \ldots, N\,, \tag{58}$$

which is just the linear system (55). The orthogonal projection operator corresponding to (57) is, of course, the identity operator: We seek a representation of $f = \mathrm{Id}_{\mathbb{R}^N} f$ with respect to the basis $\{v_1, \ldots, v_N\}$. Now we split the basis $\{v_1, \ldots, v_N\}$ into several smaller possibly overlapping subsets $\Xi_r^{N_r} := \{v_1^r, \ldots, v_{N_r}^r\} \subset \{v_1, \ldots, v_N\}$, $r = 1, \ldots, M$, such that

$$\bigcup_{r=1}^{M} \Xi_r^{N_r} = \{v_1, \ldots, v_N\}\,.$$

This union will, in general, not be disjoint, and we speak of overlapping subsets if there are at least two subsets $\Xi_r^{N_r}$, $\Xi_k^{N_k}$ with $\Xi_r^{N_r} \cap \Xi_k^{N_k} \neq \emptyset$ and $k \neq r$.

Denote the orthogonal projector from $\mathbb{R}^N$ onto $\mathrm{span}(\Xi_r^{N_r})$ by

$$P_r : \mathbb{R}^N \to \mathrm{span}\{v_1^r, \ldots, v_{N_r}^r\}\,, \quad g \mapsto P_r g\,, \tag{59}$$

i.e., $P_r = P_r \circ P_r$ and $(P_r v, w) = (v, P_r w)$ for all $v, w \in \mathbb{R}^N$. In order to compute $P_r g$, we assume again that (g, v_i), $i = 1, \ldots, N$, is known. We want to calculate the coefficient vector $y = (y_1, \ldots, y_{N_r})$ of the representation

$$P_r g = \sum_{i=1}^{N_r} y_i\, v_i^r\,.$$

Taking the inner product with $v_1^r, \dots, v_{N_r}^r$ successively leads to the linear system

$$\sum_{i=1}^{N_r} y_i \, (v_i^r, v_j^r) \stackrel{!}{=} (P_r g, v_j^r) = (g, P_r v_j^r) = (g, v_j^r)\,, \quad j = 1, \dots, N_r\,. \tag{60}$$

Clearly the matrix $\mathbf{A}_r := ((v_i^r, v_j^r))_{i,j=1,\dots,N_r}$ is a submatrix of the matrix $\mathbf{A}$ of the linear system (55).

We will now formulate the multiplicative Schwarz alternating algorithm for the solution of the trivial orthogonal projection problem $\mathrm{Id}_{\mathbb{R}^N} f = f$ in terms of the orthogonal projectors P_r. For this algorithm we prove the convergence and give information about the convergence rate. After that, we will transform this algorithm into a matrix formulation which solves (58) by solving alternatingly the problems of the type (60).

Algorithm 5.1 (Multiplicative Schwarz Alternating Algorithm)
set $f_0 := f \in \mathbb{R}^N$ *and* $s_0^f := 0$
for $n = 0, 1, 2, \dots$ *do*
  for $r = 1, \dots, M$ *do*
    calculate $s_{nM+r}^f := s_{nM+(r-1)}^f + P_r(f_{nM+(r-1)})$
    update $f_{nM+r} := f_{nM+(r-1)} - P_r(f_{nM+(r-1)})$
until $\dfrac{|((f_{(n+1)M}, v_1), \dots, (f_{(n+1)M}, v_N))^T|}{|((f, v_1), \dots, (f, v_N))^T|} \le \varepsilon$

Next, we show that the sequence of iterates $\{s_{nM}^f\}_{n\in\mathbb{N}_0}$ converges to f for $n \to \infty$. The following lemma is very helpful for the understanding of Algorithm 5.1.

Lemma 5.2 *Let the notation and the assumptions be the same as in Algorithm 5.1, and denote the orthogonal projection onto the space* $(span\{v_1^r, \dots, v_{N_r}^r\})^\perp$*, where* $r \in \{1, \dots, M\}$*, by* $Q_r : \mathbb{R}^N \to span\{v_1^r, \dots, v_{N_r}^r\})^\perp$*, i.e.,* $Q_r := Id - P_r$*. Then the following identities are valid for all* $n \in \mathbb{N}_0$*,* $r \in \{1, \dots, M\}$*:*

(i) $s_{nM+r}^f = \sum\limits_{j=1}^{r} P_j(f_{nM+(j-1)}) + \sum\limits_{l=0}^{n-1} \sum\limits_{j=1}^{M} P_j(f_{lM+(j-1)})\,,$

(ii) $f_{nM+r} = f - s_{nM+r}^f\,,$

(iii) $s_{nM+r}^f = s_{nM+(r-1)}^f + P_r(f - s_{nM+(r-1)}^f)\,,$

(iv) $f_{nM+r} = (Q_r \cdots Q_1)\,(Q_M \cdots Q_1)^n f\,,$

(v) $s_{nM+r}^f = f - (Q_r \cdots Q_1)\,(Q_M \cdots Q_1)^n f\,.$

Proof. Identities (i), (ii), and (iv) follow straightforward by induction, whereas (iii) and (iv) are simple consequences of (i), (ii), and (iv). The proof can, for example, be found in [16]. ∎

Looking at the identities (i) to (iii), we see that Algorithm 5.1 has the standard structure of an iterative algorithm: We start with $f_0 := f$, $s_0^f := 0$ and compute $s_1^f := P_1(f)$, where P_1 is an approximation of $\mathrm{Id}_{\mathbb{R}^N}$. Then we calculate the residual $f_1 := f - P_1(f)$. After that we compute $P_2(f_1) = P_2(f - s_1^f)$, where this time P_2 is used as an approximation of $\mathrm{Id}_{\mathbb{R}^N}$. Then we calculate the new iterate $s_2^f := s_1^f + P_2(f_1)$ and the new residual $f_2 := f - s_2^f = f_1 - P_2(f_1)$. This process is repeated using successively $P_3, \ldots, P_M$ as approximations of $\mathrm{Id}_{\mathbb{R}^N}$. Then the first iterative step is completed, and we start again with P_1 and proceed in the same fashion as before:

$$s^f_{nM+r} = s^f_{nM+(r-1)} + P_r(f - s^f_{nM+(r-1)}) \,.$$

In order to get the new iterate s^f_{nM+r}, we solve the problem approximately for the residual $f_{nM+(r-1)} = f - s^f_{nM+(r-1)}$ and add this solution to the old iterate $s^f_{nM+(r-1)}$. But in contrast to standard iterative algorithms the approximate solution is alternatingly computed with $P_1, \ldots, P_M$.

Identities (iv) and (v) in Lemma 5.2 are important for the convergence proof of Algorithm 5.1. Identity (v) shows that the approximation error of s^f_{nM} is given by

$$|f_{nM}| = |f - s^f_{nM}| = |(Q_M \cdots Q_1)^n f| \,. \tag{61}$$

The convergence of the residual $\{f_{nM}\}_{n\in\mathbb{N}_0}$ to zero for $n \to \infty$ follows either from the following theorem about the product of orthogonal projection operators or can be proved elementary.

Theorem 5.3 *Let $\mathcal{H}$ be a Hilbert space with inner product $(\cdot,\cdot)_{\mathcal{H}}$, let $\mathcal{U}_1, \ldots, \mathcal{U}_M$ be closed subspaces of $\mathcal{H}$, and let $Q_i : \mathcal{H} \to \mathcal{U}_i$, $i \in \{1, \ldots, M\}$, be the orthogonal projector onto $\mathcal{U}_i$. Denote by $P : \mathcal{H} \to \bigcap_{i=1}^M \mathcal{U}_i$ the orthogonal projector onto $\bigcap_{i=1}^M \mathcal{U}_i$, and define $Q : \mathcal{H} \to \mathcal{H}$ by $Q = Q_M \cdots Q_1$. Then $\{Q^n\}_{n\in\mathbb{N}_0}$ converges pointwise to P, i.e.,*

$$\lim_{n\to\infty} \|Q^n F - PF\|_{\mathcal{H}} = 0 \quad \textit{for all} \quad F \in \mathcal{H} \,.$$

Proof. This theorem is a special case of the results in [15]. ■

Corollary 5.4 *Let the notation and the assumptions be the same as in Algorithm 5.1 and Lemma 5.2. Then the sequence $\{s^f_{nM}\}_{n\in\mathbb{N}_0}$ of iterates of a vector $f \in \mathbb{R}^N$ converges to f.*

Proof. According to Lemma 5.2

$$\lim_{n\to\infty} |f - s^f_{nM}| = \lim_{n\to\infty} |(Q_M \ldots Q_1)^n f| = |Pf| \,,$$

where P is the orthogonal projector $P : \mathbb{R}^N \to \mathbb{R}^N$ onto $\bigcap_{r=1}^M \mathrm{im}(Q_r)$. But

$$\begin{aligned} \bigcap_{r=1}^M \mathrm{im}(Q_r) &= \bigcap_{r=1}^M \left(\mathrm{span}\left\{v_1^r, \ldots, v_{N_r}^r\right\}\right)^\perp \qquad (62) \\ &= \left(\sum_{r=1}^M \mathrm{span}\left\{v_1^r, \ldots, v_{N_r}^r\right\}\right)^\perp \qquad (63) \\ &= \left(\mathbb{R}^N\right)^\perp \;=\; \{0\} \,. \end{aligned}$$

Hence $P = 0$ and $\lim_{n\to\infty} |f - s^f_{nM}| = 0$. The equality of (62) and (63) follows from the following statement: Let $\mathcal{H}$ be a Hilbert space with inner product $(\cdot,\cdot)_{\mathcal{H}}$ and let $\mathcal{V}_1, \ldots, \mathcal{V}_n$ be M closed subspaces of $\mathcal{H}$. Then

$$\left(\sum_{i=1}^M \mathcal{V}_i\right)^\perp = \bigcap_{i=1}^M \mathcal{V}_i^\perp \,, \qquad \text{where} \qquad \sum_{i=1}^M \mathcal{V}_i := \mathrm{span}\left\{\bigcup_{i=1}^M \mathcal{V}_i\right\} . \tag{64}$$

The inclusion '$\subset$' is obvious, and '$\supset$' follows because $F \in \bigcap_{i=1}^{M} \mathcal{V}_i^{\perp}$ is also orthogonal to every element in $\mathrm{span}\{\bigcup_{i=1}^{M} \mathcal{V}_i\}$. ■

As mentioned before, there is also a second proof of the convergence of Algorithm 5.1, which uses only elementary results from functional analysis and yields convergence in the operator norm and an estimate of the convergence rate.

Theorem 5.5 *Let the notation and the assumptions be the same as in Algorithm 5.1 and Lemma 5.2. Then the sequence $\{s_{nM}^{f}\}_{n\in\mathbb{N}_0}$ of iterates of a vector $f \in \mathbb{R}^N$ converges to f, and the error estimate*

$$|s_{nM}^{f} - f| \leq C^n \, |f|$$

with some constant $C < 1$, independent of f, is valid.

Proof. According to Lemma 5.2, for all $n \in \mathbb{N}$

$$|f - s_{nM}^{f}| = |(Q_M \dots Q_1)^n \, f| \leq \|(Q_M \dots Q_1)\|^n \, |f| \; .$$

For simplicity of notation, we denote $Q := Q_M \dots Q_1$. The proof is complete if we can show that $\|Q\| < 1$. As all $Q_r, r \in \{1, \dots, M\}$, are orthogonal projectors $\|Q_r\| \leq 1$, and, consequently,

$$\|Q\| = \|Q_M \dots Q_1\| \leq \prod_{r=1}^{M} \|Q_r\| \leq 1 \, .$$

We show now, that the assumption $\|Q\| = 1$ leads to a contradiction. If

$$1 = \|Q\| = \sup_{g\in\mathbb{R}^n,\, |g|=1} |Qg|$$

there exists a sequence $\{g_n\}_{n\in\mathbb{N}} \subset \mathbb{R}^N$ with $|g_n| = 1$ for all $n \in \mathbb{N}$ and

$$\lim_{n\to\infty} |Qg_n| = 1 \, .$$

The space $\mathbb{R}^N$ is finite dimensional and has, therefore, a compact unit sphere. Hence $\{g_n\}_{n\in\mathbb{N}}$ has a convergent subsequence $\{g_{n_m}\}_{m\in\mathbb{N}}$, whose limit $g \in \mathbb{R}^N$ is a vector with $|g| = 1$ and

$$|Qg| = \lim_{m\to\infty} |Qg_{n_m}| = 1 \, .$$

Thus

$$|Qg| = |g| \; . \tag{65}$$

Now, we show that (65) implies that $Q_r g = g$ for all $r \in \{1, \dots, M\}$: Firstly, observe that the orthogonal projector Q_r satisfies for all $w \in \mathbb{R}^N$

$$\begin{aligned} |w|^2 &= |Q_r w + (\mathrm{Id}_{\mathbb{R}^N} - Q_r)w|^2 \\ &= |Q_r w|^2 + |(\mathrm{Id}_{\mathbb{R}^N} - Q_r)w|^2 \, , \end{aligned} \tag{66}$$

because $(Q_r w, (\mathrm{Id}_{\mathbb{R}^N} - Q_r)w) = 0$. Equation (66) shows that

$$|w| = |Q_r w| \quad \text{is equivalent to} \quad |(\mathrm{Id}_{\mathbb{R}^N} - Q_r)w| = 0 \, .$$

This means that

$$\begin{aligned} Q_r w = w \quad &\text{is equivalent to} \quad |Q_r w| = |w| \;, \\ Q_r w \neq w \quad &\text{is equivalent to} \quad |Q_r w| < |w| \;. \end{aligned} \tag{67}$$

Consider now $g \in \mathbb{R}^N$, which satisfies (65). Let $j \in \{1, \ldots, M\}$ be the smallest index for which $Q_r g \neq g$. According to (67), $|Q_j g| < |g|$, and thus,

$$\begin{aligned} |g| = |Qg| \;&=\; |Q_M \ldots Q_j g| \\ &\leq\; \|Q_M \ldots Q_{j+1}\| \, |Q_j g| \\ &<\; |g| \,, \end{aligned}$$

which is contradiction. Hence, there can be no such j and

$$Q_r g = g \quad \text{for all} \quad r \in \{1, \ldots, M\} \,. \tag{68}$$

But (68) implies that $g \in \mathrm{im}(Q_r)$ for all $r \in \{1, \ldots, M\}$:

$$g \in \bigcap_{r=1}^{M} \mathrm{im}\,(Q_r) = \bigcap_{r=1}^{M} \left(\mathrm{span}\{v_1^r, \ldots, v_{N_r}^r\}\right)^{\perp} \quad .$$

According to (64), this is equivalent to

$$g \in \left(\sum_{r=1}^{M} \mathrm{span}\{v_1^r, \ldots, v_{N-r}^r\}\right)^{\perp} = \Big(\mathrm{span}\{v_1, \ldots, v_N\}\Big)^{\perp} = (\mathbb{R}^N)^{\perp} = \{0\} \quad .$$

Thus, $g = 0$ which is a contradiction to $|g| = 1$. Consequently, the assumption $\|Q\| = 1$, is wrong. ■

Now, we come back to Algorithm 5.1 and transform it into a matrix formulation via (58) and (60). For this purpose, we need the following restriction operators $R_r : \mathbb{R}^N \to \mathbb{R}^{N_r}$, $w \mapsto R_r(w) = ((R_r(w))_1, \ldots, (R_r(w))_{N_r})^T$, and the embedding operators $I_r : \mathbb{R}^{N_r} \to \mathbb{R}^N$, $z \mapsto I_r(z) = ((I_r(z))_1, \ldots, (I_r(z))_N)^T$, corresponding to the subspaces $\mathbb{R}^{N_r}$ of the subproblems (60). They are defined by

$$\begin{aligned} (R_r(w))_i \;&:=\; w_j \text{ for the index } j \in \{1, \ldots, N\} \text{ with } v_i^r = v_j \,, \\ (I_r(z))_i \;&:=\; \begin{cases} z_j & \text{if there exists } j \in \{1, \ldots, N_r\} \text{ with } v_j^r = v_i \\ 0 & \text{else}\,. \end{cases} \end{aligned}$$

Algorithm 5.6 (matrix formulation of Algorithm 5.1)

define the matrices $\mathbf{A}_r := ((v_i^r, v_j^r))_{i,j=1,\ldots,N_r}$, $r = 1, \ldots M$

set $\tilde{f}_0 := ((f, v_1), \ldots, (f, v_N))^T$, $a_0 := (0, \ldots, 0)^T \in \mathbb{R}^N$, *where* $f \in \mathbb{R}^N$

for $n = 0, 1, 2 \ldots$ *do*

  for $r = 1, \ldots, M$ *do*

    solve $\mathbf{A}_r \, d = R_r(\tilde{f}_{nM+(r-1)})$, $d = (d_1, \ldots, d_{N_r})^T \in \mathbb{R}^{N_r}$

    update $a_{nM+r} := a_{nM+(r-1)} + I_r(d)$

    update $\tilde{f}_{nM+r} = \tilde{f}_{nM+(r-1)} - \left(\left(\sum_{i=1}^{N_r} d_i \, (v_i^r, v_k)\right)_{k=1,\ldots,N}\right)^T$

until $\dfrac{|\tilde{f}_{(n+1)M}|}{|\tilde{f}_0|} \le \varepsilon$

compute $s^f_{(n+1)M} := \sum_{i=1}^{M} (a_{(n+1)M})_i \, v_i$.

It remains to show that Algorithm 5.6 solves our initial problem $\mathbf{A}\, x = b$, where $f \in \mathbb{R}^N$ in Algorithm 5.6 satisfies $(f, v_j) = b_j$, $j = 1, \ldots, N$, and where $\mathbf{A} = ((v_i, v_j))_{i,j=1,\ldots,N}$.

Beforehand, we stress that all the computations (except the computation of $s^f_{(n+1)M}$) in Algorithm 5.6 can be performed without actually computing $v_1, \ldots, v_N \in \mathbb{R}^N$, i.e., we do not need the Cholesky factorization of $\mathbf{A}$: The matrices $\mathbf{A}_r$ are available as submatrices of $\mathbf{A}$, and the update involves a matrix vector multiplication with the matrix $((v_k, v^r_i))_{\substack{k=1,\ldots,N;\\ i=1,\ldots,N_r}}$, which is also a submatrix of $\mathbf{A}$.

Corollary 5.7 *Let the notation and the assumptions be the same as in Algorithm 5.6. Then the sequence* $\{a_{nM}\}_{n\in\mathbb{N}_0} \subset \mathbb{R}^N$ *in Algorithm 5.6 converges to the solution* $x \in \mathbb{R}^N$ *of the linear system* $\mathbf{A}\, x = b$*, where* $\mathbf{A} = ((v_i, v_j))_{i,j=1,\ldots,N}$ *and* $b = ((f, v_1), \ldots, (f, v_N))^T$.

Proof. According to Theorem 5.5 and equation (58) and (60), we know that $\{s^f_{nM}\}_{n\in\mathbb{N}_0}$

$$s^f_{nM} := \sum_{i=1}^{N} (a_{nM})_i \, v_i \,,$$

converges to $f = \sum_{i=1}^{N} x_i \, v_i$, where x is the solution of $\mathbf{A}\, x = b$. As $\{v_1, \ldots, v_N\}$ is a basis of $\mathbb{R}^N$ this implies that

$$\lim_{n\to\infty} (a_{nM})_i = x_i \quad \text{for} \quad i = 1, \ldots, N \,.$$

This proves the convergence. ■

5.2 Concluding Remarks on the Numerical Implementation for Spline Approximation

Finally, we want to give some comments concerning the implementation of Algorithm 5.6 for the solution of the linear equation systems given by Problems 4.3, 4.6, and 4.7. We choose the space $\mathcal{H}(\{A_n\};\Omega)$ or $h^{(3)}(\{A_n\};\Omega)$ among those discussed in Subsection 4.3, so that the matrix entries and the representers are available as elementary functions. This enables us to evaluate a spline and to compute a matrix entry with small computational effort. In an implementation of Algorithm 5.6, we will generate only the small matrices $\mathbf{A}_r$ in advance, compute, for example, their Cholesky factorization in a preprocessing step, and keep the matrices of the Cholesky factorizations of the $\mathbf{A}_r$ in the memory. The other matrix entries of $\mathbf{A}$, which will be needed for the update (computation of the new residual), are generated while the update is performed. The update is the time-consuming task, whereas the smaller equation systems can now be solved extremely fast. If fast multipole methods (fast summation technics) are available for the type of kernel, which determines the matrix entries, the update can be accelerated. The matrix entries in Problems 4.3, 4.6, and 4.7 depend on a grid of points on Ω. Consequently, a splitting of the matrix can correspond to a subdivision of the sphere into (overlapping) subsets. This explains, why we speak of a domain decomposition method.

Numerical tests of the multiplicative Schwarz alternating algorithm for vectorial spline interpolation or smoothing will be presented in a forthcoming paper based on wind field data of the 'Forstliche Versuchsanstalt (FVA) des Landes Rheinland-Pfalz, Trippstadt'. The multiplicative Schwarz alternating algorithm was tested in [1], [13], and [16] for scalar spline interpolation and smoothing in Sobolev spaces $\mathcal{H}(\{A_n\};\overline{\Omega_R^{\mathrm{ext}}})$ (in the context of geopotential determination) and showed a very good numerical performance.

Acknowledgments:

The authors acknowledge the support by 'Stiftung Innovation für Rheinland-Pfalz'.

References

[1] P. Baumann. *Regularization of Inverse Problems in Satellite Geodesy by Wavelet Methods with Orbital Data Given on Closed Surfaces.* Diploma Thesis, Geomathematics Group, Department of Mathematics, University of Kaiserslautern, (2002).

[2] M. Bayer, S. Beth, W. Freeden. *Geophysical Field Modelling by Multiresolution Analysis.* Acta Geodaetica et Geophysica Hungarica, 33, pp. 289-319, (1998).

[3] R.K. Beatson, S. Billings, W.A. Light. *Fast Solution of the Radial Basis Function Interpolation Equations: Domain Decomposition Methods.* SIAM, J. Sci. Comput., Vol. 22, No. 5, pp. 1717-1740, (2000).

[4] S. Beth. *Multiscale Approximation by Vector Radial Basis Functions on the Sphere.* Doctoral Thesis, Geomathematics Group, Department of Mathematics, University of Kaiserslautern, Shaker-Verlag, 2000.

[5] *Proceedings of the X-th International Symposium of Domain Decomposition Methods (for Partial Differential Equations).* SIAM Philadelphia (held anually since 1987).

[6] H.W. Engl, M. Hanke, A. Neubauer. *Regularization of Inverse Problems.* Kluwer Academic Publishers, (1996).

[7] W. Freeden. *On Integral Formulas of the (Unit) Sphere and Their Application to Numerial Computations of Integrals.* Computing, 25, pp. 131-146, (1980).

[8] W. Freeden. *On Spherical Spline Interpolation and Approximation.* Math. Meth. in the Appl. Sci., 3, pp. 551-575, (1981).

[9] W. Freeden. *Multiscale Modelling of Spaceborne Geodata.* B.G. Teubner, Stuttgart, Leipzig, (1999).

[10] W. Freeden, T. Gervens, M. Schreiner. *Constructive Approximation on the Sphere (With Applications to Geomathematics).* Oxford Science Publications, Clarendon Press, Oxford, (1998).

[11] W. Frommer, H. Schwandt. *A Unified Representation and Theory of Algebraic Additive Schwarz and Multisplitting Methods.* SIAM J. Matrix Anal. Appl., Vol. 18, No. 4, pp. 893-912, (1997).

[12] M. Griebel, P. Oswald. *On the abstract theory of additive and multiplicative Schwarz algorithms.* Numer. Math., 70, pp. 163-180, (1995).

[13] M. Gutting. *Multiscale Gravitational Field Modelling from Oblique Derivatives.* Diploma Thesis, Geomathematics Group, Department of Mathematics, University of Kaiserslautern, (2002).

[14] G. Hämmerlin, K.-H. Hoffmann. *Numerische Mathematik.* Springer-Verlag, Berlin, Heidelberg, New York, (1992).

[15] I. Halperin. *The Product of Projection Operators.* Acta Sci. Math., 23, pp. 96-99, (1992).

[16] K. Hesse. *Domain Decomposition Methods in Multiscale Geopotential Determination from SST and SGG.* Doctoral Thesis, Geomathematics Group, Department of Mathematics, University of Kaiserslautern, (2002).

[17] A. Kirsch. *An Introduction to the Mathematical Theory of Inverse Problems.* Springer-Verlag, New York, (1996).

[18] S. Levitus. *Climatological Atlas of the World Ocean.* NOAA Professional Paper 13, Geophysical Fluid Dynamics Laboratory, Rockville, Maryland, (1982).

[19] P.L. Lions. *On the Schwarz Alternating Method I.* In: Proceedings of the 1st International Symposium on Domain Decomposition Methods for Partial Differential Equations, Paris, January 1987, R. Glowinski, G.H. Golub, G. Meurant, J. Periaux (eds.), SIAM Philadelphia, pp.1-42, (1988).

[20] P.L. Lions. *On the Schwarz Alternating Method II: Stochastic Interpretation and Order Properties.* In: Proceedings of the 2nd International Symposium on Domain Decomposition Methods, Los Angeles, California, January 1998, T.F. Chan, R. Glowinski, J. Periaux, O.B. Widlund (eds.), SIAM Philadelphia, pp. 47-70 (1989).

[21] P.L. Lions. *On the Schwarz Altenating Method III: A Variant for Nonoverlapping Subdomains.* In: Proceedings of the 3rd International Symposium on Domain Decomposition Methods for Partial Differential Equations, Houston, Texas, March 1989, T.F. Chan, R. Glowinski, J. Periaux, O.B. Widlund (eds.), SIAM Philadelphia, pp. 202-223, (1990).

[22] R.S. Nerem, B.D. Tapley, C.K. Shum. *An General Ocean Circulation Model Determined in a Simultaneous Solution With the Earth's Gravity Field.* International Association of Geodesy Symposia, No. 104, H. Sünkel, T. Baker (eds.), 158-166, (1990).

[23] R.S. Nerem, C.J. Koblinsky. *The Geoid and Ocean Circulation.* In: Geoid and Its Geophysical Interpretations, P. Vanicek, N.T. Christou (eds.), CRC Press, pp. 321-338, (1994).

[24] J. Pedlovsky. *Geophysical Fluid Dynamics.* Springer, (1979).

[25] H.A. Schwarz. *Gesammelte Mathematische Abhandlungen*, Vol. 2, Springer Verlag Berlin, pp. 133-134, (1890).

[26] C. Wunsch, E.M. Gaposchkin. *On Using Satellite Gradiometry to Determine the General Circulation of the Ocean With Application to Geoid Improvement*. Review of Geophysics and Space Physics, Vol. 18, pp. 725-745, (1980).

[27] J. Xu. *Iterative Methods by Space Decomposition and Subspace Correction*. SIAM Review, Vol. 34, No. 4, pp. 581-613, (1992).

[28] H. Yamabe. *On an Extension of the Helly's Theorem*. Osaka Mathematical Journal, Vol. 2 (1), pp. 15-17, (1950).